Solutions to Homework #10

Problems from Pages 451-452 (Section 8.5)

8. To find the radius of convergence we will carry out the Ratio test with $a_n = \dfrac{x^n}{n \cdot 3^n}$. First we will set up and simplify the ratio.

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)\cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{x^n} = \frac{n}{3(n+1)} x .$$

The limit of the absolute value of this ratio (as $n \to \infty$) is $\frac{1}{3}|x|$. Setting this to be less than 1 and solving for the absolute value of x gives that the radius of convergence of the power series is 3.

To determine the interval of convergence we must examine the convergence of the power series at the endpoints of the interval ($x = -3$ and $x = 3$). We do this by plugging the endpoints into the power series and then determining the convergence or divergence of the resulting infinite series using one of the convergence or divergence tests covered earlier in Chapter 8.

***x* = −3:** The power series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. This alternating series converges conditionally by the Alternating Series Test because $\frac{1}{n+1} < \frac{1}{n}$ and the limit of $\frac{1}{n}$ is zero as $n \to \infty$. Therefore $x = -3$ is included in the interval of convergence.

***x* = 3:** The power series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. This is a p-series with $p = 1$ (so it diverges). Therefore $x = 3$ is not included in the interval of convergence.

The interval of convergence of the power series is $[-3, 3)$.

12. To find the radius of convergence we will carry out the Ratio test with $a_n = \dfrac{(-1)^n}{(2n)!} x^{2n}$. First we set up and simplify the ratio.

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}}{(2n+2)!} \cdot x^{2n+2} \cdot \frac{(2n)!}{(-1)^n} \cdot \frac{1}{x^{2n}} = \frac{-x^2}{(2n+2)\cdot(2n+1)}.$$

The limit of the absolute value of this ratio (as $n \to \infty$) is zero. This means that the limit of the absolute value of the ratio is less than one no matter what x is. So, the power series converges for all values of x. This means that the radius of convergence of the power series is infinite and the interval of convergence of the power series is the set of all real numbers.

20. The radius of convergence of the power series $\sum_{n=0}^{\infty} c_n \cdot x^n$ is at least 4 but smaller than 6. This means that for x in the interval $[-4, 4)$, the power series definitely converges, but when $x < -6$ or when $x \geq 6$, the power series diverges.

(a) $x = 1$. The series converges.

(b) $x = 8$. The series diverges.

(c) $x = -3$. The series converges.

(d) $x = -9$. The series diverges.

Problems from Pages 456-458 (Section 8.6)

4. In this problem we will make use of the familiar power series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Replacing x by x^4 and multiplying by 3 gives the power series that we need:

$$\frac{3}{1-x^4} = \sum_{n=0}^{\infty} 3 \cdot \left(x^4\right)^n = \sum_{n=0}^{\infty} 3 \cdot x^{4n}.$$

To find the interval of convergence we will begin by finding the radius of convergence. To do this we will use the Ratio test with $a_n = 3 \cdot x^{4n}$. Fist we create and simplify the ratio:

$$\frac{a_{n+1}}{a_n} = \frac{3 \cdot x^{4(n+1)}}{3 \cdot x^{4n}} = x^4.$$

The limit of the absolute value of this ratio as $n \to \infty$ is $|x|^4$. Forcing this to be less than 1 gives that the radius of convergence is 1.

The endpoints of the interval of convergence are $x = -1$ and $x = +1$. To determine whether or not these points are included in the interval of convergence we must plug them into the power series and examine the convergence or divergence of the resulting series.

***x* = 1:** The power series becomes $\sum_{n=0}^{\infty} 3$ which diverges by the n^{th} term test.

***x* = –1:** The power series becomes $\sum_{n=0}^{\infty} 3$ which diverges by the n^{th} term test.

Neither endpoint is included in the interval of convergence, which is (–1, 1).

8. In this problem we will make use of the familiar power series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n .$$

Rearranging the give formula for $f(x)$ gives us that:

$$f(x) = \frac{x}{4x+1} = x \cdot \frac{1}{1-(-4x)} .$$

Replacing x in the existing power series by $-4x$ and multiplying the result by x gives the power series for $f(x)$:

$$f(x) = x \cdot \sum_{n=0}^{\infty} (-4x)^n = \sum_{n=0}^{\infty} (-1)^n \cdot 4^n \cdot x^{n+1} .$$

To find the interval of convergence we will begin by finding the radius of convergence. To do this we will carry out the Ratio test with $a_n = (-1)^n \cdot 4^n \cdot x^{n+1}$. First, we set up and simplify the ratio:

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1} \cdot 4^{n+1} \cdot x^{n+2}}{(-1)^n \cdot 4^n \cdot x^{n+1}} = -4x .$$

The limit of the absolute value of this ratio (as $n \to \infty$) is $4|x|$. Forcing this to be less than one and solving for the absolute value of x gives that the radius of convergence is 0.25.

To determine the interval of convergence we must plug the endpoints of the interval (x = –0.25 and x = 0.25) into the power series for $f(x)$ and determine whether or not the resulting infinite series are convergent or divergent.

$x = -0.25$: The power series becomes $\sum_{n=0}^{\infty} 1$ which diverges by the n^{th} term test.

$x = 0.25$: The power series becomes $\sum_{n=0}^{\infty} (-1)^n$ which diverges by the n^{th} term test.

Neither endpoint is included in the interval of convergence, which is $\left(\frac{-1}{4}, \frac{1}{4}\right)$.

12. We will begin by using the technique of partial fractions to break up the given function $f(x)$:

$$f(x) = \frac{7x-1}{3x^2+2x-1} = \frac{A}{x+1} + \frac{B}{3x-1}.$$

Adding the two fractions and equating coefficients of powers of x in the numerators gives the two equations specifying A and B:

$$3A + B = 7 \qquad \text{and} \qquad -A + B = -1.$$

Solving these equations gives $A = 2$ and $B = 1$ so that:

$$f(x) = \frac{2}{x+1} + \frac{1}{3x-1} = \frac{2}{1-(-x)} + \frac{-1}{1-(3x)}.$$

We will now make use of the familiar power series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

substituting $-x$ and $3x$ for x to give:

$$f(x) = \frac{2}{1-(-x)} + \frac{-1}{1-(3x)} = \sum_{n=0}^{\infty} 2\cdot(-1)^n \cdot x^n - \sum_{n=0}^{\infty} 3^n \cdot x^n = \sum_{n=0}^{\infty}\left[2\cdot(-1)^n - 3^n\right]\cdot x^n.$$

To find the interval of convergence we will consider the intervals of convergence of the two power series that were combined in $f(x)$. The interval of convergence of the sum is the intersection of these two intervals of convergence.

The interval of convergence of $\sum_{n=0}^{\infty} 2 \cdot (-1)^n \cdot x^n$ is $(-1, 1)$. The interval of convergence of $-\sum_{n=0}^{\infty} 3^n \cdot x^n$ is $\left(\frac{-1}{3}, \frac{1}{3}\right)$. The intersection of these two intervals is $\left(\frac{-1}{3}, \frac{1}{3}\right)$.

Problems from Pages 469-471 (Section 8.7)

2. The graph supplied on page 469 of the textbook shows that near $x = a = 1$, the function is increasing and concave up. This means that $f'(1) > 0$ and $f''(1) > 0$.

The first Taylor series that is supplied in the problem has -0.8 as the coefficient of $(x - 1)$. As $f'(1)$ is the coefficient of $(x - 1)$ in the Taylor series, and we know from that graph that $f'(1) > 0$, the supplied series cannot be the Taylor series of $f(x)$ with $a = 1$.

The graph supplied on page 469 of the textbook shows that near $x = a = 2$, the function is increasing and concave down. This means that $f'(2) > 0$ and $f''(2) < 0$.

The second Taylor series that is supplied in the problem has $+1.5$ as the coefficient of $(x - 2)^2$. As $\frac{f''(2)}{2!}$ is the coefficient of $(x - 2)^2$ in the Taylor series, and we know from the graph that $f''(2) < 0$, the supplied series cannot be the Taylor series of $f(x)$ with $a = 2$.

4. We will use the fundamental definition of the Taylor series as:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

to find the Taylor series in this function using $a = 4$ and $f^{(n)}(4) = \frac{(-1)^n \cdot n!}{3^n \cdot (n+1)}$.

Substituting these directly into the above definition gives the Taylor series for $f(x)$:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(4)}{n!}(x-4)^n=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot n!}{3^n\cdot(n+1)}\cdot\frac{1}{n!}\cdot(x-4)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n\cdot(n+1)}\cdot(x-4)^n.$$

To find the **radius of convergence** of this Taylor series we will carry out the Ratio test with $a_n=\frac{(-1)^n}{3^n\cdot(n+1)}(x-4)^n$. First we form and simplify the ratio:

$$\frac{a_{n+1}}{a_n}=\frac{(-1)^{n+1}}{3^{n+1}\cdot(n+2)}(x-4)^{n+1}\cdot\frac{3^n\cdot(n+1)}{(-1)^n}\cdot\frac{1}{(x-4)^n}=\frac{-(n+1)}{3(n+2)}(x-4).$$

Next, take the limit of the absolute value of this ratio as $n\to\infty$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\tfrac{1}{3}\cdot|x-4|<1.$$

The limit is forced to be less than 1 so that the Ratio test will guarantee convergence. Solving for the absolute value of $x - 4$ gives that the radius of convergence is equal to 3.

14. We will use the fundamental definition of the Taylor series as:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

to find the Taylor series for $f(x) = \ln(x)$ based at $a = 2$. To do this we will first calculate the first few derivatives of $f(x)$ and evaluate these at $a = 2$. Doing this:

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	$\ln(x)$	$\ln(2)$
1	$\frac{1}{x}$	$\frac{1}{2}$
2	$\frac{-1}{x^2}$	$\frac{-1}{4}$
3	$\frac{2}{x^3}$	$\frac{2}{8}$

Looking at the entries in the middle column of the above table, the pattern that emerges is that for $n > 1$,

$$f^{(n)}(x)=\frac{(-1)^{n+1}\cdot(n-1)!}{x^n}.$$

Plugging $x = a = 2$ into this and substituting this into the fundamental definition of the Taylor series gives:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (n-1)!}{2^n \cdot n!}(x-2)^n = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 2^n}(x-2)^n .$$

32. The Maclaurin series (Taylor series with $a = 0$) for $\cos(x)$ is:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} .$$

Replacing x by $2x$ in this will give the Maclaurin series for $\cos(2x)$:

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n}{(2n)!} x^{2n} .$$

Finally, multiplying this result by x will give the Maclaurin series for $f(x)$:

$$f(x) = x \cdot \cos(2x) = x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n}{(2n)!} x^{2n+1} .$$