SOLUTIONS

Recitation Handout 17: Radius and Interval of Convergence

Interval of Convergence

The interval of convergence of a power series: $\sum_{n=0}^{\infty} c_n \cdot (x-a)^n$ is the interval of x-values that can be plugged into the power series to give a convergent series.

The center of the interval of convergence is always the anchor point of the power series, a.

Radius of Convergence

The radius of convergence is half of the length of the interval of convergence. If the radius of convergence is R then the interval of convergence will include the open interval:

$$(a-R, a+R).$$

Finding the Radius of Convergence

To find the radius of convergence, R, you use the Ratio Test.

Step 1: Let $a_n = c_n \cdot (x-a)^n$ and $a_{n+1} = c_{n+1} \cdot (x-a)^{n+1}$.

Step 2: Simplify the ratio $\frac{a_{n+1}}{a_n} = \frac{c_{n+1} \cdot (x-a)^{n+1}}{c_n \cdot (x-a)^n} = \frac{c_{n+1}}{c_n} \cdot (x-a)$.

Step 3: Compute the limit of the absolute value of this ratio as $n \to \infty$.

Step 4: Interpret the result using the table below.

Limit of absolute value of ratio as $n \to \infty$.	Radius of convergence, R.
Zero.	Infinite. The power series converges for all values of x.
$N \cdot \lvert x-a \rvert$, where N is a finite, positive number.	$R = \frac{1}{N}$. The interval of convergence includes $\left(a-\frac{1}{N}, a+\frac{1}{N}\right)$ and possibly the end-points $x = a - \frac{1}{N}$ and $x = a + \frac{1}{N}$.
Infinity.	Zero. The power series converges at $x = a$ and nowhere else.

Are the end-points in the Interval of Convergence?

Each of the two end-points ($x = a - R$ and $x = a + R$) may or may not be part of the interval of convergence. To determine whether the end-points are in the interval of convergence, you have to plug them into the power series (one at a time) to get an infinite series. You then use a convergence test to determine whether or not the infinite series converges or diverges. If the infinite series converges, then the end-point that you plugged into the power series is in the interval of convergence. Otherwise, the end-point is not in the interval of convergence.

If you use the ratio test at each end-point you usually get an inconclusive test so it is best to try a different convergence test when investigating the end-points of the interval of convergence.

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(a) Find the radius of convergence and interval of convergence for: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot (x-1)^n$

$$a_n = \frac{(-1)^{n+1}}{n}(x-1)^n \qquad a_{n+1} = \frac{(-1)^{n+2}}{n+1}(x-1)^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+2}}{n+1} \cdot \frac{n}{(-1)^{n+1}} \cdot \frac{(x-1)^{n+1}}{(x-1)^n}$$

$$= \frac{(-1)\,n}{n+1} \cdot (x-1)$$

Limit of absolute value of ratio = $\lim_{n\to\infty} \left| \frac{(-1)\,n}{n+1} \cdot (x-1) \right|$

$$= |x-1|.$$

Radius of convergence = 1.

RADIUS OF CONVERGENCE: 1.

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Find the radius of convergence and interval of convergence for: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot (x-1)^n$

End points: $1 - 1 = 0$ and $1 + 1 = 2$.

Convergence or divergence at first end-point:

Plugging $x = 0$ into the power series gives the infinite series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot (-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$$

Since the harmonic series $\left(\sum_{n=1}^{\infty} \frac{1}{n} \right)$ diverges, the power series diverges when $x=0$ is plugged in. So, $x=0$ is <u>not</u> part of the interval of convergence.

Convergence or divergence at second end-point:

Plugging $x=2$ into the power series gives the infinite series:

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot (1)^n = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$. This is the alternating harmonic series which converges. Set $a_n = 1/n$. Then $0 \leq \frac{1}{n+1} < \frac{1}{n}$ means that $0 \leq a_{n+1} < a_n$ and $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$. So, by the Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$ converges. This means that $x=2$ <u>is</u> part of the interval of convergence.

INTERVAL OF CONVERGENCE: $(0, 2]$

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(b) Find the radius of convergence and interval of convergence for: $\sum_{n=0}^{\infty} 2^{2n} \cdot x^{2n}$

$$a_n = 2^{2n} \cdot x^{2n} \qquad a_{n+1} = 2^{2n+2} \cdot x^{2n+2}$$

$$\frac{a_{n+1}}{a_n} = \frac{2^{2n+2}}{2^{2n}} \cdot \frac{x^{2n+2}}{x^{2n}} = 2^2 \cdot x^2$$

$$= 4x^2$$

Limit of absolute value of ratio = $\lim_{n \to \infty} \left| 4x^2 \right| = 4|x|^2.$

For convergence using the ratio test,

$$4|x|^2 < 1$$

$$|x|^2 < 1/4$$

$$|x| < 1/2$$

RADIUS OF CONVERGENCE: 1/2.

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Find the radius of convergence and interval of convergence for: $\sum_{n=0}^{\infty} 2^{2n} \cdot x^{2n}$

End points: $0 - 1/2 = -1/2$ and $0 + 1/2 = 1/2$

Convergence or divergence at first end-point:

Plugging $x = -1/2$ into the power series gives the infinite series:

$$\sum_{n=0}^{\infty} 2^{2n} \cdot \left(\frac{-1}{2}\right)^{2n} = \sum_{n=0}^{\infty} \frac{2^{2n}}{2^{2n}} = \sum_{n=0}^{\infty} 1.$$

This infinite series diverges. Set $a_n = 1$. Then $\lim_{n \to \infty} a_n = 1 \neq 0$ so the infinite series diverges using the n^{th} term test for divergence.

Therefore, $x = -1/2$ is not part of the interval of convergence.

Convergence or divergence at second end-point:

Plugging $x = 1/2$ into the power series gives the infinite series:

$$\sum_{n=0}^{\infty} 2^{2n} \cdot \left(\frac{1}{2}\right)^{2n} = \sum_{n=0}^{\infty} \frac{2^{2n}}{2^{2n}} = \sum_{n=0}^{\infty} 1.$$

This infinite series diverges. Set $a_n = 1$. Then $\lim_{n \to \infty} a_n = 1 \neq 0$ so the infinite series diverges by the n^{th} term test for divergence.

Therefore, $x = 1/2$ is not part of the interval of convergence either.

INTERVAL OF CONVERGENCE: $(-1/2, 1/2)$

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(c) Find the radius of convergence and interval of convergence for: $\sum_{n=1}^{\infty} \frac{4^n}{n} \cdot (x-3)^n$

$$a_n = \frac{4^n}{n}(x-3)^n \qquad a_{n+1} = \frac{4^{n+1}}{n+1}(x-3)^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}}{4^n} \cdot \frac{n}{n+1} \cdot \frac{(x-3)^{n+1}}{(x-3)^n}$$

$$= \frac{4n}{n+1}(x-3)$$

Limit of absolute value of ratio = $\lim_{n\to\infty} \left| \frac{4n}{n+1}(x-3) \right|$

$$= 4 \cdot |x-3|$$

Radius of convergence $= 1/4$

RADIUS OF CONVERGENCE: 1/4

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Find the radius of convergence and interval of convergence for: $\sum_{n=1}^{\infty} \frac{4^n}{n} \cdot (x-3)^n$

End points: $3 - \frac{1}{4} = \frac{11}{4}$ and $3 + \frac{1}{4} = \frac{13}{4}$

Convergence or divergence at first end-point:

Substituting $x = 11/4$ into the power series gives the infinite series:

$$\sum_{n=1}^{\infty} \frac{4^n}{n} \left(\frac{11}{4} - 3\right)^n = \sum_{n=1}^{\infty} \frac{4^n}{n} \cdot \left(\frac{-1}{4}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This infinite series converges. Set $a_n = 1/n$. Then $0 \leq \frac{1}{n+1} < \frac{1}{n}$ so that $0 \leq a_{n+1} < a_n$. In addition, $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$. So, by the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. This means that $x = 11/4$ is in the interval of convergence.

Convergence or divergence at second end-point:

Substituting $x = 13/4$ into the power series gives the infinite series:

$$\sum_{n=1}^{\infty} \frac{4^n}{n} \cdot \left(\frac{13}{4} - 3\right)^n = \sum_{n=1}^{\infty} \frac{4^n}{n} \cdot \left(\frac{1}{4}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}.$$

This infinite series diverges. Let $f(x) = 1/x$ then when $x \geq 1$, $f(x) = 1/x > 0$ and $f'(x) = -1/x^2 < 0$.

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{a \to \infty} \int_1^a \frac{1}{x}\,dx = \lim_{a \to \infty} \left[\ln|x|\right]_1^a$$

$$= \lim_{a \to \infty} \ln(a) = +\infty.$$

By the integral test, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges so the end-point $x = 13/4$ is not part of the interval of convergence.

INTERVAL OF CONVERGENCE: $[11/4, 13/4)$

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(d) Find the radius of convergence and interval of convergence for:

$$1+2\cdot(x+5)+\frac{4!}{(2!)^2}\cdot(x+5)^2+\frac{6!}{(3!)^2}\cdot(x+5)^3+\frac{8!}{(4!)^2}\cdot(x+5)^4+\ldots$$

$$a_n = \frac{(2n)!}{(n!)^2}\cdot(x+5)^n \qquad a_{n+1} = \frac{(2n+2)!}{((n+1)!)^2}(x+5)^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(2n)!}\cdot\frac{(n!)^2}{((n+1)!)^2}\cdot\frac{(x+5)^{n+1}}{(x+5)^n}$$

$$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)}\cdot(x+5)$$

$$= \frac{4n+2}{n+1}(x+5)$$

Limit of absolute value of ratio = $\lim\limits_{n\to\infty}\left|\frac{4n+2}{n+1}(x+5)\right|$

$$= 4\cdot|x+5|$$

So the radius of convergence is 1/4.

RADIUS OF CONVERGENCE: 1/4.

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Find the radius of convergence and interval of convergence for:

$$1+2\cdot(x+5)+\frac{4!}{(2!)^2}\cdot(x+5)^2+\frac{6!}{(3!)^2}\cdot(x+5)^3+\frac{8!}{(4!)^2}\cdot(x+5)^4+\ldots$$

End points: $-5-\frac{1}{4} = \frac{-21}{4}$ and $-5+1/4 = \frac{-19}{4}$

Convergence or divergence at first end-point:

When you plug $x=-\frac{21}{4}$ into the power series you get the infinite series:

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}\left(-\frac{21}{4}+5\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n\cdot(2n)!}{(n!)^2\cdot 4^n}.$$

This infinite series converges. Set $a_n = \frac{(2n)!}{(n!)^2\cdot 4^n}$. Then

$$a_{n+1} = \frac{(2n+1)\,a_n}{(2n+2)}$$

so that $0 \le a_{n+1} \le a_n$. It is possible (but not easy) to obtain that $\lim_{n\to\infty} a_n = 0$ so that the infinite series converges by the Alternating Series Test.

Convergence or divergence at second end-point:

When you plug $x=-19/4$ into the power series you get the infinite series:

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}\left(\frac{-19}{4}+5\right)^n = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2\cdot 4^n}.$$

This infinite series diverges, although this is very difficult to demonstrate rigorously. To get an idea of whether the series converges or diverges, try the following MAPLE commands:

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f:= n -> sum((2*k)!/((k!)^2 * 4^k);
limit(f(n), n = infinity);
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INTERVAL OF CONVERGENCE: $[-5.25, -4.75)$

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ANSWERS:

(a) Radius of convergence = 1. Interval of convergence is $(0, 2]$.

(b) Radius of convergence = 0.5. Interval of convergence is $(-0.5, 0.5)$.

(c) Radius of convergence = 0.25. Interval of convergence is $[2.75, 3.25)$.

(d) Radius of convergence = 0.25. Interval of convergence is $[-5.25, -4.75)$.