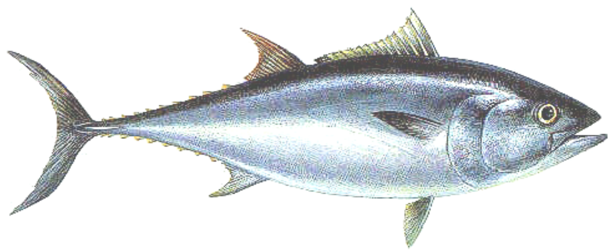
Solutions for Recitation Handout 16: Taylor Series and the Atlantic Bluefin Tuna

The giant bluefin tuna (*Thunnus thynnus*) is the largest bony fish known to science. This fish can grow to a length of eleven feet and weigh up to 1500 pounds[1]. The bluefin tuna is a remarkably strong fish, and is able to retract its fins and eyes to make it more streamlined. Bluefin tuna have been observed to swim at speeds of up to 55 miles per hour[2].



Bluefin tuna are valued as a food source, especially as sushi and sashimi. A large tuna in excellent condition sell for more than $100,000 when auctioned at *Tsukiji*, the main fish market in Tokyo, Japan[3]. Bluefin have been commercially fished in the western Atlantic since the 1960's, with the industry firmly established by the 1980's[4]. Bluefin fishing has become such a lucrative business that commercial tuna fishermen routinely use spotter aircraft to find the fish[5].

Contrary to many perceptions[6], most tuna fisheries, although heavily exploited are not seriously over fished[7]. This is not the case for bluefin tuna. Bluefin have been heavily overf ished for at least two decades[8]. Based on studies conducted by the International Convention for the Conservation of Atlantic Tuna (ICCAT) and the National Research Council (NRC), the breeding population of Atlantic bluefin tuna fell from approximately 235,000 in 1975 to less than 40,000 in the late 1990's.

In 1998, ICCAT proposed an historic plan to limit catch sizes of bluefin tuna to allow the population to recover. In this handout, you investigate the effects of the ICCAT plan on the population of Atlantic bluefin tuna.

---

[1] Source: New England Aquarium (www.neaq.org). The largest bluefin tuna on record was caught by Ken Fraser in Canada during 1979. The fish that Fraser caught weighed 677 kg (1497 pounds).
[2] Source: World Wildlife Fund. (www.panda.org).
[3] Source: NASA and the Smithsonian Institution Ocean Planet Project (seawifs.gsfc.nasa.gov).
[4] Source: National Academy of Sciences, National Research Council. *An Assessment of Atlantic Bluefin Tuna*. Washington, DC: National Academy Press, 1994.
[5] Source: World Wildlife Fund.
[6] For example, see: Cole, J. N. "The Vanishing Tuna." *Atlantic Monthly*, Volume 239, p. 50. (Dec. 1976)
[7] Source: Environmental Protection Agency, Revised Final Enviromental Impact Statement to accompany Fisheries Management Plan for Highly Migratory Species, 1999.
[8] Source: Buck, Eugene. "Atlantic Bluefin Tuna: International Management of a Shared Resource." The National Council for Science and the Environment, Washington DC, 1995.

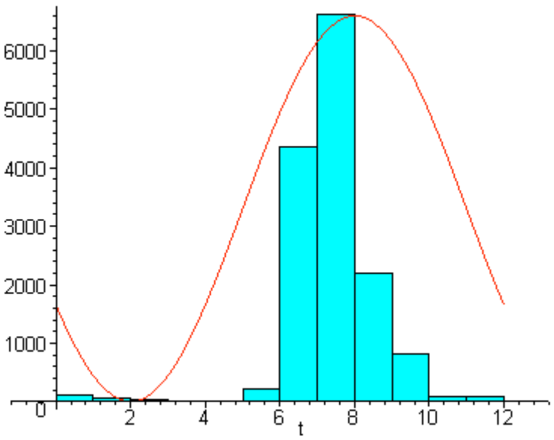


Figure 1: Number of bluefin tuna caught by month, 1998.

The commercial bluefin fishing season runs from June 1 to May 31, or until the quota has been reached[9]. As shown in the histogram (Figure 1, left) most fish are caught between July and October, with relatively few fish harvested during the rest of the year[10]. Regulations governing the size of tuna that can be caught[11] mean that it is mainly sexually mature adults that are harvested.

The curve in Figure 1 is a rough model for the rate at which tuna caught (in tuna caught per month)[12]. The equation for the curve is:

$$Rate = 3.3 - 3.3\cos\left(\tfrac{\pi}{6}(t-2)\right)$$

where the rate of capture has the units of thousands of tuna per month, and t represents the number of months since January 1998 (when the ICCAT plan went into effect).

Studies are now underway to study the bluefin tuna, and particularly whether or not the plan proposed by ICCAT in 1991 will enable the bluefin tuna population to recover or not. These studies involve attaching electronic tags to tuna. These tags are monitored by satellites and the information collected about the tuna relayed to scientists in the US and Canada[13]. Some of the data collected from these studies is shown in Figure 2. The rates of change given in Figure 2 are for sexually mature bluefin tuna and do not include juveniles.

The information given in Figure 2 (see next page) has been synthesized into a mathematical model for the Atlantic bluefin tuna population. This model can be represented by a differential equation and an initial condition:

$$\frac{dP}{dt} = 0.0875 \cdot P(t) - 3.3 + 3.3 \cdot \cos\left(\tfrac{\pi}{6}(t-2)\right) \qquad \text{and} \qquad P(0) = 40,$$

where $P(t)$ is the size of the sexually mature Atlantic bluefin tuna population (in units of thousands of tuna) and t is the number of months since January 1998.

---

[9] Source: U.S. Department of Commerce, National Oceanic and Atmospheric Administration, National Marine Fisheries Service. "Small Entity Compliance Guide for the Consolidated Regulations of Atlantic Tuna, Swordfish, Sharks and Billfish." 1999.

[10] Source of data: Personal communication from the National Marine Fisheries Service, Fisheries Statistics and Economics Division, Silver Spring, MD.

[11] Source: U.S. Department of Commerce, National Oceanic and Atmospheric Administration.

[12] Source of data: National Marine Fisheries Service, Fisheries Statistics and Economics Division, and Roberta Holland. "Bluefin tuna prices hit rock bottom." *Boston Business Journal* (September 11, 1998).

[13] Some of this research is being carried out locally by Dr. Molly Lutcavage a marine biologist at the New England Aquarium.

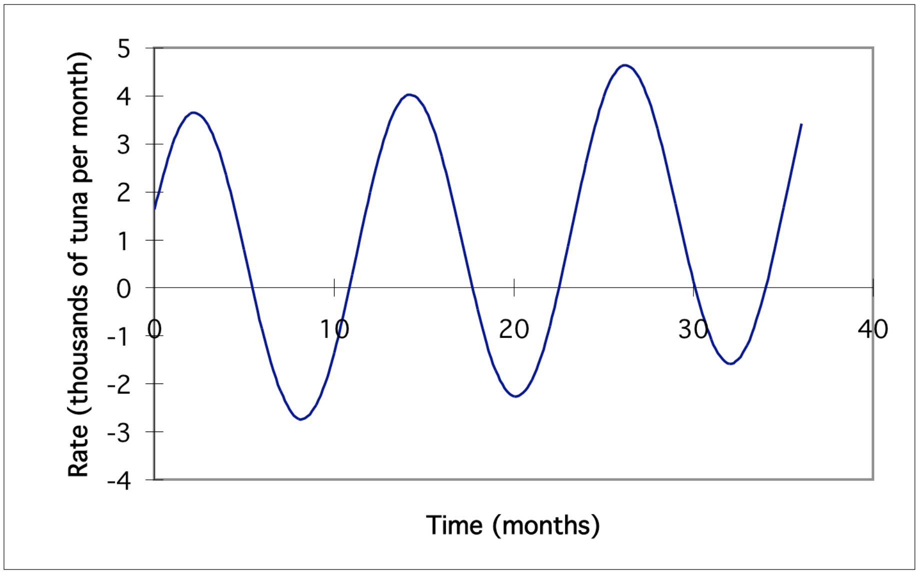
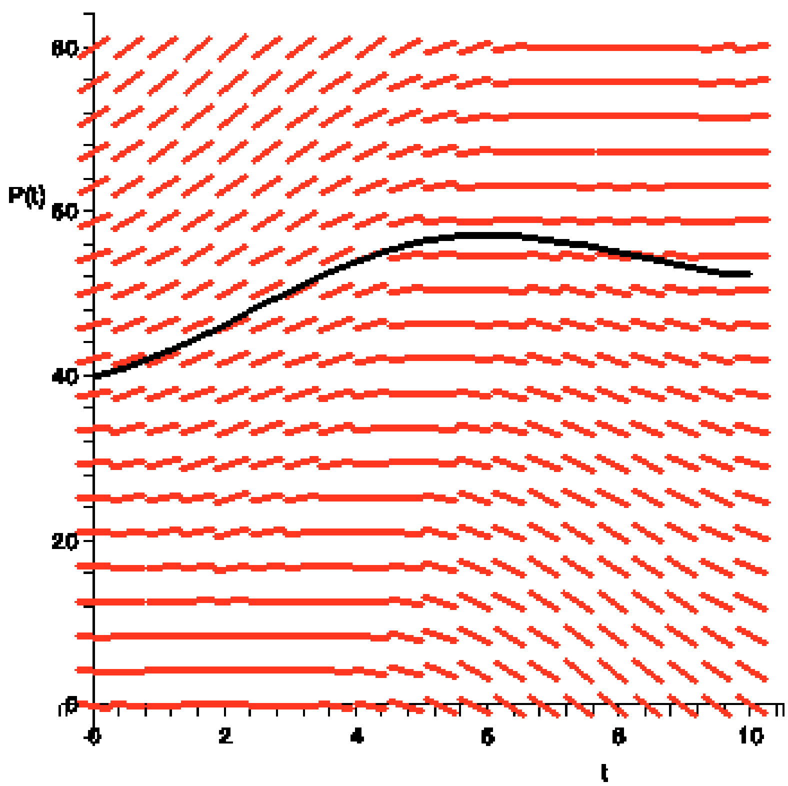


Figure 2: Rate of change of bluefin tuna breeding population (thousands of tuna per month) 1998-2001. (Source: International Commission for Conservation of Atlantic Tuna).

1. Use the axes provided below, the information that $P(0) = 40$ and Figure 2 to sketch a plausible graph for the function $P(t)$. Based on your sketch, did the ICCAT help the Atlantic bluefin tuna population to increase their numbers?



There are several ways to come up with the graph of $P(t)$ versus t shown above. The method that was used here was to use the differential equation:

$$\frac{dP}{dt} = 0.0875 \cdot P(t) - 3.3 + 3.3 \cdot \cos\left(\tfrac{\pi}{6}(t-2)\right)$$

and the grid provided to draw a slope field. Once the slope field has been drawn, a solution curve starting at the initial point (0, 40) can be sketched.

2. Use the differential equation and initial condition to find the formula for the tangent line approximation to the curve $y = P(t)$ that is based at $t = 0$. What does the tangent line suggest about the effectiveness of the ICCAT plan that was intended to help the Atlantic bluefin tuna?

When $t = 0$ we have that $P(0) = 40$. Substituting these figures into the differential equation will give the slope of the tangent line:

$$\left.\frac{dP}{dt}\right|_{t=0} = 0.0875 \cdot (40) - 3.3 + 3.3 \cdot \cos\left(\tfrac{\pi}{6}(-2)\right) = 1.85.$$

The intercept of the tangent line can be calculated by substituting $m = 1.85$, $P = 40$ and $t = 0$ into the equation for a straight line:

$$P = m \cdot t + b$$

to calculate the intercept, b. Doing this gives $b = 40$ and the equation of the tangent line is:

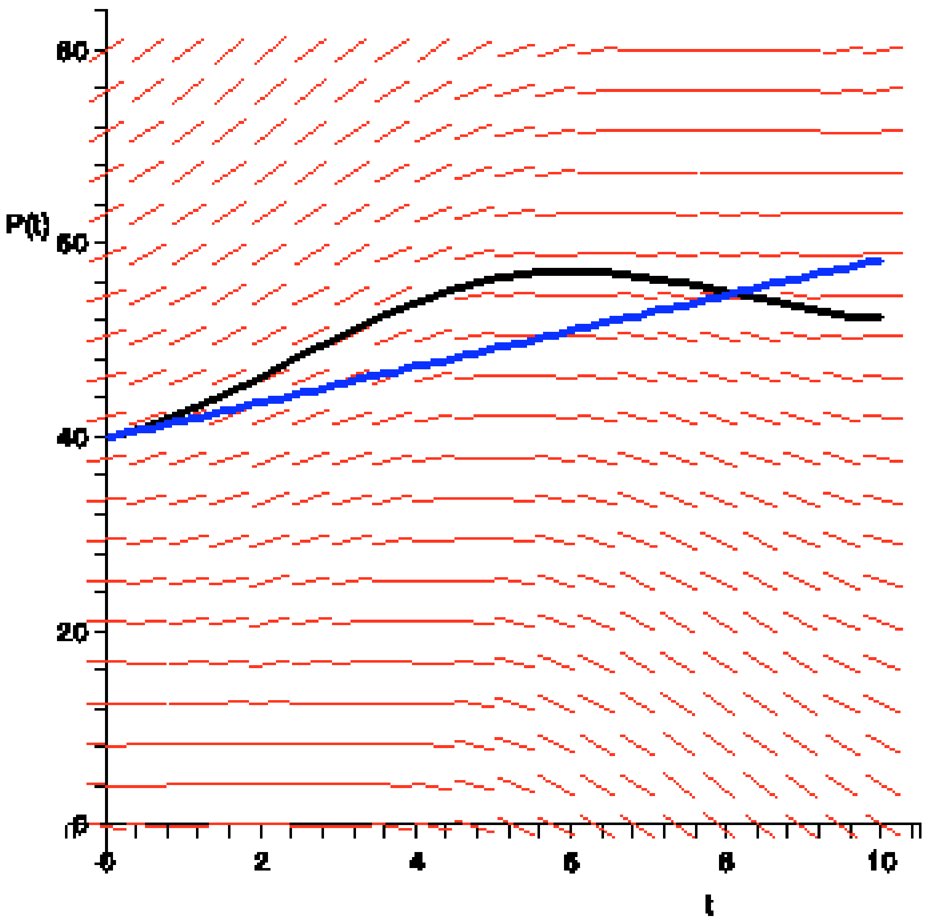
$$P = 1.85 \cdot t + 40.$$

3. Using the same set of axes that you used in Question 1, sketch the tangent line. For which values of t does the tangent line seem to do a decent job of approximating the function $P(t)$?

A graph showing both the tangent line and the sketch of the solution curve for the differential equation is shown on the next page. The solution curve is shown in black and the tangent line is shown in blue.

The solution curve and the tangent line start together and are very difficult to distinguish from each other initially. It is over this interval that the two graphs are very difficult to distinguish that we say the blue tangent line is doing a good job of approximating the black solution curve.

Based on inspection of the graph on the next page, the tangent line and the solution curve are difficult to tell apart between $t = 0$ and $t = 0.66$ so it is not unreasonable to say that over the interval [0, 0.66] the tangent line does a good job of approximating the solution curve.



4. Find the formula for the second degree Taylor polynomial approximation of the function $P(t)$ that is based at zero. (That is, the anchor point is $a = 0$.)

The second degree Taylor polynomial approximation for the function $P(t)$ is:

$$P(0) + P'(0) \cdot t + \frac{P''(0)}{2} \cdot t^2.$$

We already know that $P(0) = 40$ and $P'(0) = 1.85$. In order to be able to write down the second degree Taylor polynomial we need to be able to calculate the second derivative of the function $P(t)$. If we take the derivative of the differential equation:

$$\frac{dP}{dt} = 0.0875 \cdot P(t) - 3.3 + 3.3 \cdot \cos\left(\tfrac{\pi}{6}(t-2)\right)$$

then we will obtain:

$$\frac{d^2P}{dt^2} = 0.0875 \cdot P'(t) - 3.3 \cdot \sin\left(\tfrac{\pi}{6}(t-2)\right) \cdot \tfrac{\pi}{6}.$$

Substituting $P'(0) = 1.85$ and $t = 0$ into this expression gives the second derivative of $P(t)$ evaluated at $t = 0$:

$$\left.\frac{d^2P}{dt^2}\right|_{t=0} = 0.0875 \cdot (1.85) - 3.3 \cdot \sin\left(\tfrac{\pi}{6}(-2)\right) \cdot \tfrac{\pi}{6} = 1.685259475.$$

So, the formula for the second degree Taylor polynomial that approximates the function $P(t)$ with $a = 0$ is given by:

$$40 + 1.85 \cdot t + 0.829 \cdot t^2.$$

5. Find the formula for the degree three Taylor polynomial approximation of the function $P(t)$ that is based at zero. (That is, the anchor point is $a = 0$.)

The degree three Taylor polynomial approximation of $P(t)$ with $a = 0$ is given by:

$$P(0) + P'(0) \cdot t + \frac{P''(0)}{2} \cdot t^2 + \frac{P'''(0)}{3!} \cdot t^3.$$

We know the values of all of the constants in this formula with the exception of the third derivative of $P(t)$ evaluated at $t = 0$. To find this we will differentiate the expression for $\frac{d^2P}{dt^2}$ that was obtained in Question 4. Doing this gives:

$$\frac{d^3P}{dt^3} = 0.0875 \cdot P''(t) - 3.3 \cdot \cos\left(\tfrac{\pi}{6}(t-2)\right) \cdot \left(\tfrac{\pi}{6}\right)^2.$$

Substituting $P''(0) = 1.685259475$ and $t = 0$ into this expression allows us to find the value of $P'''(0)$:

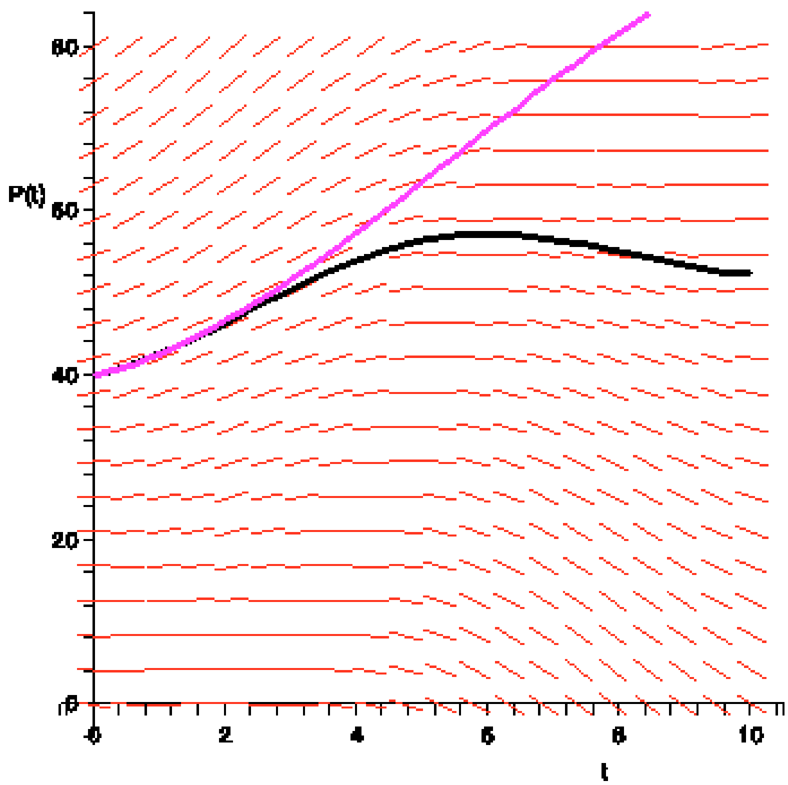
$$\left.\frac{d^3P}{dt^3}\right|_{t=0} = 0.0875 \cdot (1.685259475) - 3.3 \cdot \cos\left(\tfrac{\pi}{6}(-2)\right) \cdot \left(\tfrac{\pi}{6}\right)^2 = -0.3072591643.$$

The degree three Taylor polynomial approximating the function $P(t)$ with $a = 0$ is given by the formula:

$$40 + 1.85 \cdot t + 0.829 \cdot t^2 - 0.051 \cdot t^3.$$

6. Use the axes given below to sketch your version of the $P(t)$ function from Question 1. Then graph your third degree Taylor polynomial on your calculator and transfer the graph to the set of axes given below. Based on your sketch, over what set of t-values does the Taylor polynomial do a decent job of matching the graph of $P(t)$?

A graph showing both the initial sketch of $P(t)$ (in black) and the degree three Taylor polynomial approximating $P(t)$ (in purple) is given below.



The interval over which the purple curve and the black curve are difficult to distinguish is now quite a bit longer than the interval from Question 3 (which was [0, 0.66]). Based on a visual inspection of the diagram shown above, the two curves are very close to each other between $t = 0$ and $t = 2.66$. It is not unreasonable to say that the degree three Taylor polynomial does a good job of approximating the function $P(t)$ over the interval [0, 2.66].

7. Why do you think a marine biologist or fisheries researcher might want to create a Taylor polynomial to approximate the function $P(t)$? What would be the main advantage of using a Taylor polynomial with a lot of terms in it?

A biologist would want to know $P(t)$ so that he or she could predict the size of the tuna population as t increases and determine whether or not the ICCAT regulations were helping the tuna population to replenish itself.

The differential equation defining $P(t)$ is very difficult (perhaps impossible) to solve to find a formula that will give the exact values of $P(t)$.

A Taylor polynomial is a practical way to calculate approximate values for $P(t)$. The advantages of using a Taylor polynomial with many terms to do this are:

1. The values of $P(t)$ calculated will be more accurate.

2. The interval of t-values over which $P(t)$ closely matches the solution curve will be longer, so the biologist will be able to make predictions for larger values of t, corresponding to further into the future.