SOLUTIONS

Math 122 **Fall 2008**

Recitation Handout 15: Practice Using Convergence Tests

Determine the convergence or divergence of each of the following series. In each case, demonstrate that your answer is correct in a step-by-step fashion using *an appropriate convergence test*. Be sure to explicitly state which convergence test you have used and show that it can be used with the series you are working on. Be careful to show all of your work and how the convergence test justifies your answer.

(a) Does the following series converge or diverge? $$\sum_{n=1}^{\infty} \frac{n^2}{1+4n^2}$$

CONVERGENCE TEST USED:

n^{th} Term Test for Divergence.

STEP-BY-STEP JUSTIFICATION:

The n^{th} term of the infinite series is:

$$a_n = \frac{n^2}{1+4n^2}.$$

Calculating the limit as $n \to \infty$ gives:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{1+4n^2} = \frac{1}{4} \neq 0.$$

Since $\lim_{n\to\infty} a_n \neq 0$, the n^{th} Term Test for Divergence gives that this infinite series diverges.

FINAL CONCLUSION: CONVERGES DIVERGES

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(b) Does the following series converge or diverge? $$\sum_{n=1}^{\infty} \frac{10^n \cdot \sqrt{n}}{(n+1)!}$$

CONVERGENCE TEST USED:

Ratio Test.

STEP-BY-STEP JUSTIFICATION:

Here $a_n = \dfrac{10^n \cdot \sqrt{n}}{(n+1)!}$ so $a_{n+1} = \dfrac{10^{n+1}\sqrt{n+1}}{(n+2)!}$.

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}\sqrt{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{10^n\sqrt{n}} = \frac{10\sqrt{n+1}}{(n+2)\sqrt{n}}$$

Taking the limit of the ratio as $n \to \infty$:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{10}{n+2} = 0.$$

Since $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, the

Ratio Test gives that the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{10^n \cdot \sqrt{n}}{(n+1)!}$ converges.

FINAL CONCLUSION: CONVERGES DIVERGES

(c) Does the following series converge or diverge? $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

CONVERGENCE TEST USED:

Comparison Test.

STEP-BY-STEP JUSTIFICATION:

Note that $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a p-series with $p = 3/2 > 1$, so convergent. Note that when $n > 1$:

$$n^2 < n^2 + 1$$

$$\frac{n^2}{\sqrt{n}} < \frac{n^2+1}{\sqrt{n}}$$

Simplifying: $$n^{3/2} < \frac{n^2+1}{\sqrt{n}}$$

$$\frac{1}{n^{3/2}} > \frac{\sqrt{n}}{n^2+1} \geq 0$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent and $0 \leq \frac{\sqrt{n}}{n^2+1} \leq \frac{1}{n^{3/2}}$, the Comparison Test gives that $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ also converges.

FINAL CONCLUSION: CONVERGES DIVERGES

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(d) Does the following series converge or diverge? $\sum_{n=1}^{\infty} \frac{1+\sin(n)}{n^2}$

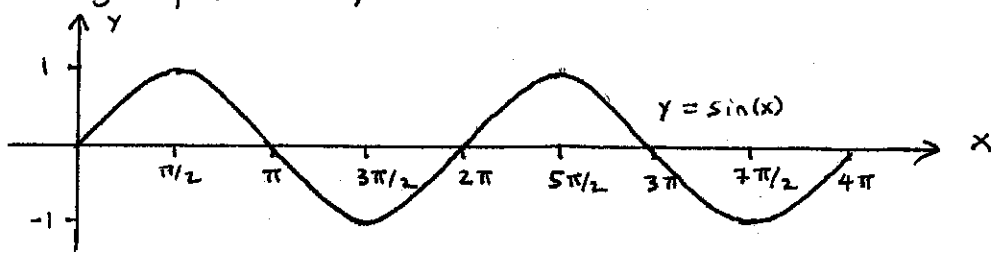
CONVERGENCE TEST USED:

Comparison Test.

STEP-BY-STEP JUSTIFICATION:

Note that $\sum_{n=1}^{\infty} \frac{2}{n^2}$ is a constant multiple of the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $(p=2)$, so $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges.

The graph of $y = \sin(x)$ is shown below.



The graph shows that: $-1 \leq \sin(n) \leq 1$.

Therefore:

$$0 \leq 1 + \sin(n) \leq 2$$

$$0 \leq \frac{1+\sin(n)}{n^2} \leq \frac{2}{n^2}$$

Since $\frac{1+\sin(n)}{n^2} \leq \frac{2}{n^2}$ and $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges, the Comparison Test gives that $\sum_{n=1}^{\infty} \frac{1+\sin(n)}{n^2}$ also converges.

FINAL CONCLUSION: CONVERGES DIVERGES

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(e) Does the following series converge or diverge? $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots$

CONVERGENCE TEST USED:

Integral test. (Comparison also good.)

STEP-BY-STEP JUSTIFICATION:

To do the integral test, note that the infinite series can be written in Σ-notation as:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} .$$

We will use the integral test with $f(x)=\frac{1}{\sqrt{x}}$. Note that for $x>0$, $f(x)=1/\sqrt{x}$ is positive and decreasing so the integral test can be used.

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{a\to\infty}\int_1^a \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{a\to\infty}\Big[2\sqrt{x}\Big]_1^a$$

$$= \lim_{a\to\infty} 2\sqrt{a} - 2$$

$$= +\infty$$

As the improper integral $\int_1^\infty \frac{1}{\sqrt{x}}dx$ diverges, so too, the infinite series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges.

FINAL CONCLUSION: CONVERGES DIVERGES

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(f) Does the following series converge or diverge? $\displaystyle\sum_{n=1}^{100} \frac{2n}{1+n^2}$

CONVERGENCE TEST USED:

None.

STEP-BY-STEP JUSTIFICATION:

The series is a finite series. You can tell because the upper limit of summation is a finite number (and not the ∞ symbol).

All finite series converge, so this must be a convergent series.

FINAL CONCLUSION: CONVERGES DIVERGES

ANSWERS:

(a)	Diverge.	**(b)**	Converge.	**(c)**	Converge.	**(d)**	Converge.
(e)	Diverge.	**(f)**	Converge.				