SOLUTIONS

Math 122 **Fall 2008**

Handout 15: Review Problems for the Cumulative Final Exam

The topics that will be covered on Final Exam are as follows.

- Integration formulas.
- U-substitution.
- Integration by parts.
- Integration using trigonometric formulas and identities.
- Trigonometric substitution.
- Partial fractions.
- Integration tricks such as polynomial long division and completing the square.
- Approximating integrals (Riemann sums, trapezoid and midpoint rules, Simpson's rule).
- Conditions under which different integration methods produce over and under estimates.
- Error estimates for trapezoid, midpoint and Simpson's rules.
- Improper integrals.
- Areas between curves.
- Calculating volumes (non-rotation) using integrals.
- Volumes of revolution – disk method.
- Volumes of revolution – washer method.
- Volumes of revolution – shell method.
- Arc length.
- Using integrals to calculate masses.
- Center of mass.
- Using integrals to calculate work in physics.
- Using integrals to calculate hydrostatic force.
- Euler's method.
- Slope fields.
- Equilibrium solutions.
- Separation of variables.
- Integrating factors.
- Second order, homogeneous differential equations with constant coefficients.
- Method of Undetermined Coefficients.
- Calculating formulas for partial sums (e.g. for telescoping series).
- Convergence of infinite series by definition (limit of partial sums).
- Geometric series (finite) and their applications.
- Geometric series (infinite) and their applications.
- n^{th} Term test for divergence.
- Integral test.
- Ratio test.
- Comparison test (compare with p-series or infinite geometric series).
- Alternating series test.
- Absolute versus conditional convergence for alternating series.
- Estimating the sum of an alternating series to a given level of accuracy.
- Summing a finite series with a calculator.
- Finding a formula for the Taylor series of $f(x)$ with center a from the definition.
- Finding a formula for the Taylor series of $f(x)$ with center a by modifying an existing series.
- Radius of convergence of a power series or Taylor series.
- Interval of convergence of a power series or Taylor series.
- Accuracy of Taylor polynomial approximations for functions.
- Parametric equations for circles, line segments, ellipses and other curves in the plane.

- Calculating tangent lines for curves defined by parametric equations.
- Finding arc lengths for curves defined by parametric equations.
- Sketching curves defined by polar equations.
- Finding equations for tangent lines when curves are defined by polar equations.

This (roughly) covers the end of Chapter 5, all of Chapters 6-8 and the first half of Chapter 9 of the textbook, together with additional topics (such as differential equations).

1. Find the Taylor series of the function $f(x) = \dfrac{x^2}{\sqrt{2+x}}$ centered at $a = 0$.

We will use the binomial theorem:

$$(1+x)^p = 1 + p\cdot x + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$$

Note that:

$$f(x) = \frac{x^2}{\sqrt{2+x}} = x^2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{1+\frac{1}{2}x}}$$

$$= x^2 \cdot \frac{1}{\sqrt{2}} \cdot \left(1 + \tfrac{1}{2}x\right)^{-1/2}$$

$$= x^2 \cdot \frac{1}{\sqrt{2}} \cdot \left(1 - \frac{1}{2}\cdot\frac{1}{2}x + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2!}\left(\tfrac{1}{2}x\right)^2 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!}\left(\tfrac{1}{2}x\right)^3 + \dots\right)$$

$$= \frac{1}{\sqrt{2}}x^2 - \frac{1}{2^2\sqrt{2}}x^3 + \frac{3}{2^5\sqrt{2}}x^4 - \frac{15}{2^7\cdot(3)\cdot\sqrt{2}}x^5 + \dots$$

SOLUTIONS

2. Consider the function $f(x) = x^{\frac{3}{4}}$.

(a) Find the Taylor polynomial of degree 3 for $f(x)$ centered at $a = 16$.

$$f(x) = x^{3/4} \qquad f(a) = 16^{3/4} = 8$$

$$f'(x) = \frac{3}{4} x^{-1/4} \qquad f'(a) = 3/8$$

$$f''(x) = \frac{-3}{16} x^{-5/4} \qquad f''(a) = \frac{-3}{512}$$

$$f'''(x) = \frac{15}{64} x^{-9/4} \qquad f'''(a) = \frac{15}{32768}$$

$$P_3(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$= 8 + \frac{3}{8}(x-16) - \frac{3}{1024}(x-16)^2 + \frac{5}{65536}(x-16)^3$$

(b) Use your answer from Part (a) to estimate the value of $17^{\frac{3}{4}}$.

Plug $x = 17$ into $P_3(x)$ from above. This gives:

$$17^{3/4} \approx P_3(17) = 8.372146606$$

(c) Find a reasonable estimate for the error in your approximation from Part (b).

According to the Taylor Remainder Theorem,

$$|f(b) - P_3(b)| \leq \frac{M}{4!}(b-a)^4$$

where M is the maximum value of $|f^{(4)}(x)|$ over the interval $[16, 17]$. Here $a = 16$ and $b = 17$.

$f^{(4)}(x) = \frac{-135}{256} x^{-13/4}$. Graphing the absolute value of this on a calculator and finding the maximum gives $M = 135/2097152$, so the estimate for the error is:

$$\text{Error} \leq \frac{\frac{135}{2097152}}{4!}(17-16)^4 \approx 2.682209 \times 10^{-6}.$$

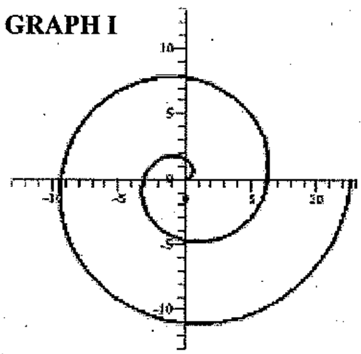
3. A flying ladybug lands in a spot of wet paint at the origin on the xy-plane. The ladybug then walks away from the origin tracing out a path described by the equations

$$x(t) = t \cdot \cos(t) \qquad y(t) = t \cdot \sin(t) \qquad t \geq 0.$$

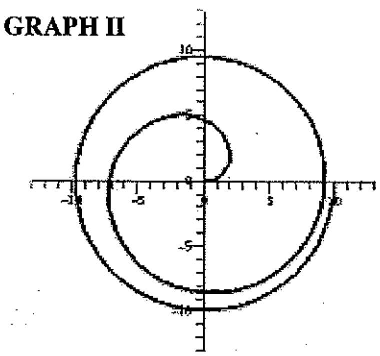
t is measured in seconds and x, y are measured in millimeters.

(a) Circle the diagram that does the best job of representing the path traced out by the ladybug over the interval $0 \leq t \leq 4\pi$.

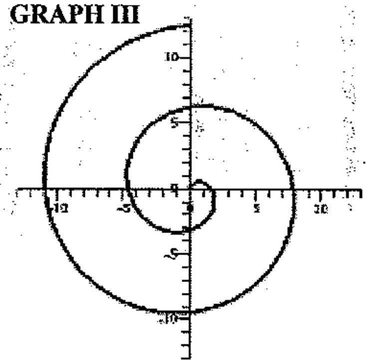
GRAPH I



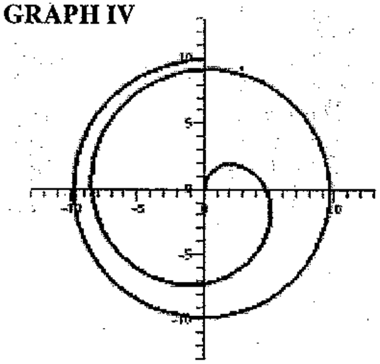
GRAPH II



GRAPH III



GRAPH IV



(b) Circle the integral that gives the *exact* length of the path that the ladybug travels during the first 2 seconds.

$$\int_0^2 \sqrt{(t \cdot \sin(t))^2 + (t \cdot \cos(t))^2}\,dt$$

$$\int_0^2 \sqrt{(-t \cdot \sin(t))^2 + (t \cdot \cos(t))^2}\,dt$$

$$\boxed{\int_0^2 \sqrt{(\cos(t) - t \cdot \sin(t))^2 + (\sin(t) + t \cdot \cos(t))^2}\,dt}$$

$$\int_0^2 \sqrt{1 + 2\sin(t)\cos(t) + t^2}\,dt$$

$$\int_0^2 \sqrt{1 + t^2 \cdot (\sin^2(t) - \cos^2(t))}\,dt$$

$$\int_0^2 \sqrt{1 + (\cos(t) - \sin(t))^2}\,dt$$

Continued on the next page.

(c) Calculate the speed of the ladybug when $t = 2\pi$. Give at least three decimal places in your answer. Remember to include appropriate units with your answer.

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(\cos(t) - t\cdot\sin(t))^2 + (\sin(t) + t\cdot\cos(t))^2}$$

Plugging $t = 2\pi$ into this gives:

$$\text{Speed} = \sqrt{1^2 + 4\pi^2} \approx 6.362265132$$

(d) Find the equation of the tangent line to the ladybug's path at $t = \frac{7\pi}{2}$.

The point on the curve corresponding to $t = 7\pi/2$ has:

$$x(7\pi/2) = 7\pi/2 \cdot \cos(7\pi/2) = 0$$

$$y(7\pi/2) = 7\pi/2 \cdot \sin(7\pi/2) = -7\pi/2.$$

Slope of tangent line:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin(t) + t\cdot\cos(t)}{\cos(t) - t\cdot\sin(t)}$$

Plugging $t = 7\pi/2$ into this gives:

$$\left.\frac{dy}{dx}\right|_{t=7\pi/2} = \frac{-1 + 0}{0 + 7\pi/2} = \frac{-2}{7\pi}$$

Equation of Tangent Line:

$$y + \frac{7\pi}{2} = \frac{-2}{7\pi}(x - 0)$$

4. Determine the convergence or divergence of each of the following series. In each case, CIRCLE either CONVERGES or DIVERGES.

In each case, *demonstrate that your answer is correct* step-by-step using *an appropriate convergence test.* Be sure to explicitly state which convergence test you have used. Be careful to show how the convergence test justifies your answer. If you do not justify your answer, you will get zero credit, even if you circle the correct response.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5+(-1)^n}{n\cdot\sqrt{n}}$ CONVERGES DIVERGES

JUSTIFICATION: We will use the comparison test.

Initial Guess: $5+(-1)^n$ is always less than 6 and $n\cdot\sqrt{n} = n^{3/2}$ so this series will be dominated by $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^{3/2}}$, a p-series with $p = 3/2 > 1$ so it converges. I would guess that $\displaystyle\sum_{n=1}^{\infty} \frac{5+(-1)^n}{n\cdot\sqrt{n}}$ also converges.

Formal Comparison: $0 < 5+(-1)^n < 6$

$$0 < \frac{5+(-1)^n}{n\cdot\sqrt{n}} < \frac{6}{n^{3/2}}$$

As $\displaystyle\sum_{n=1}^{\infty} 6/n^{3/2}$ converges (it is a p-series with $p > 1$), the Comparison test gives that $\displaystyle\sum_{n=1}^{\infty} \frac{5+(-1)^n}{n\sqrt{n}}$ also converges.

SOLUTIONS

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \cdot (\ln(n))^4}$ CONVERGES DIVERGES

JUSTIFICATION: We will use the Integral Test with $f(x) = \dfrac{1}{x \cdot (\ln(x))^4}$.

First we must verify that the Integral test applies for $x > 2$.

(I) $\underline{f(x) > 0}$ when $x > 2$, $(\ln(x))^4 > 0$ so

$$f(x) = \frac{1}{(+)(+)} = +.$$

(II) $\underline{f'(x) < 0}$

$$f'(x) = \frac{-\left[(\ln(x))^4 + x \cdot 4 \cdot (\ln(x))^3 \cdot \frac{1}{x}\right]}{x^2 \cdot (\ln(x))^8}$$

Now the numerator is negative when $x > 2$ and the denominator is positive when $x > 0$ so $f'(x) < 0$.

<u>Improper Integral:</u>

$$\int_2^{\infty} \frac{1}{x \cdot (\ln(x))^4}\,dx = \lim_{a \to \infty} \int_2^{a} \frac{1}{x \cdot (\ln(x))^4}\,dx$$

Let $u = \ln(x)$ so that $dx = x \cdot du$, and:

$$= \lim_{a \to \infty} \int_{\ln(2)}^{\ln(a)} u^{-4}\,du$$

$$= \lim_{a \to \infty} \frac{-1}{3}\left(\frac{1}{(\ln(a))^3} - \frac{1}{(\ln(2))^3}\right)$$

$$= \frac{1}{3 \cdot (\ln(2))^3}$$

As the improper integral converges, the Integral Test gives that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \cdot (\ln(n))^4}$ converges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}{n!}$ CONVERGES DIVERGES

JUSTIFICATION: We will use the Ratio Test. (Note that if you can recognize that $a_n = \dfrac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}{n!} = 2^n$ then you could also use the n^{th} term test for divergence.)

$$\frac{a_{n+1}}{a_n} = \frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)(2n+2)}{(n+1)!} \cdot \frac{n!}{2\cdot 4\cdot 6\cdot \ldots (2n)}$$

$$= \frac{(2n+2)}{(n+1)} = 2.$$

So, $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 2 > 1$ so the Ratio test gives that the series $\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}{n!}$ diverges.

SOLUTIONS

5. Consider the alternating series: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n^2}{10^n}$.

(a) Does this series converge absolutely, converge conditionally or diverge?

This series converges absolutely, which we will demonstrate using the Ratio test.

Performing the Ratio Test with $a_n = \frac{(-1)^{n-1} \cdot n^2}{10^n}$:

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^n \cdot (n+1)^2}{10^{n+1}} \cdot \frac{10^n}{(-1)^{n-1} \cdot n^2} = \frac{-1}{10} \cdot \frac{(n+1)^2}{n^2}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{-1}{10} \cdot \frac{n^2 + 2n + 1}{n^2} \right| = 1/10.$$

As $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{10} < 1$, the Ratio Test gives that the series $\sum_{n=1}^{\infty} \frac{n^2}{10^n}$ converges, so the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n^2}{10^n}$ converges absolutely.

Continued on the next page.

(b) What is the smallest value of N needed to ensure that $\sum_{n=1}^{N} \frac{(-1)^{n-1} \cdot n^2}{10^n}$ approximates the value of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n^2}{10^n}$ with an error of less than 0.0001?

As this series converges, the Alternating Series Estimation Theorem applies. Set $a_n = \frac{n^2}{10^n}$. Then:

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n^2}{10^n} - \sum_{n=1}^{N} \frac{(-1)^{n-1} \cdot n^2}{10^n} \right| < a_{N+1} = \frac{(N+1)^2}{10^{N+1}}$$

To find N that will ensure that the error is less than 0.0001, we must solve:

$$\frac{(N+1)^2}{10^{N+1}} < 0.0001.$$

Doing this with a table on a calculator gives:

$$N \geqslant 5.$$

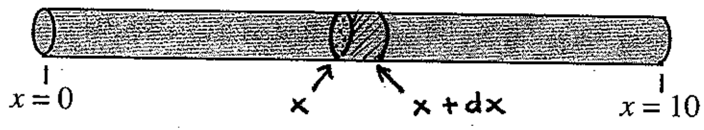
So, the smallest value of N that will lead to an error less than 0.0001 is $N = 5$.

(c) Approximate the value of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n^2}{10^n}$ with an error of less than 0.0001.

y1 = (-1) ^ (x - 1) * x ^ 2 / 10 ^ x

sum(seq(y1(K), K, 1, 5)) = 0.06765.

6. A carbon rod of length 10 cm is placed on the x-axis with its left endpoint at the origin.



The density of the carbon rod is given by the function: $\delta(x) = a + bx$ in units of grams per centimeter (g/cm). In this formula, a and b are both positive constants.

(a) Calculate the mass of the carbon rod. Your answer may include the constants a and b. Include appropriate units with your answer.

Mass located between x and $x + dx$ $= \delta(x) \cdot dx$

$= (a + bx) \cdot dx$

Total mass $= \int_0^{10} (a + bx) \cdot dx$

$= \left[ax + \frac{1}{2} bx^2 \right]_0^{10} = 10a + 50b.$ grams.

(b) Calculate the x-coordinate of the *center of mass* of the carbon rod. Your answer may include the constants a and b.

$$\text{Center of mass} = \frac{\int_0^{10} x \cdot \left\{ \begin{matrix} \text{Mass located between} \\ x \text{ and } x + dx \end{matrix} \right\}}{\text{Total mass}}$$

$$= \frac{\int_0^{10} x \cdot (a + bx) \cdot dx}{10a + 50b}$$

$$= \frac{\left[\frac{1}{2} ax^2 + \frac{1}{3} bx^3 \right]_0^{10}}{10a + 50b}$$

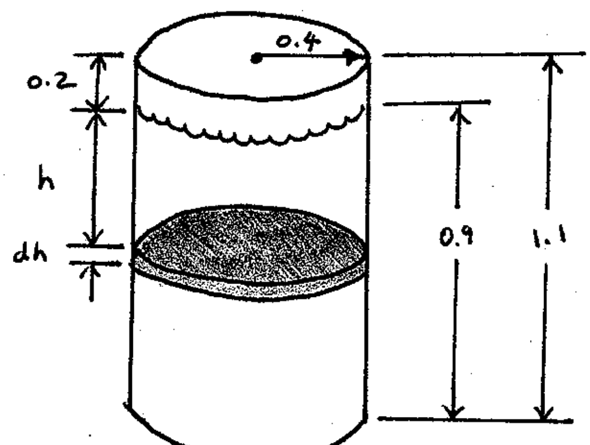
$$= \frac{50a + \frac{1000}{3} b}{10a + 50b}$$ cm from $x = 0$ end.

SOLUTIONS

7. A cylindrical urn, standing upright, contains hot chocolate. The top of the urn has a *diameter* of 0.8 m. The urn has a *height* of 1.1 m and is filled to a *depth* of 0.9 m. The density of the hot chocolate at a depth of h meters below the surface is given by the function:

$$\delta(h) = 1 + A \cdot h \qquad \text{kg/m}^3,$$

where A is a positive constant. Find the total work required to pump the hot chocolate to the top rim of the urn. Remember to include appropriate units in your answer. Note that in SI units, the constant for gravity is $g = 9.8$ m/s^2.



The diagram shows the physical situation. We will start by calculating the work that must be done to lift the shaded slice of hot chocolate to the top of the urn, and then use an integral to calculate the total work.

$$\begin{aligned}\text{Work for slice} &= (\text{Force on slice}) \cdot (\text{Distance moved}) \\ &= (\text{mass})(9.8) \cdot (h + 0.2) \\ &= (\text{volume})(\text{density})(9.8)(h + 0.2) \\ &= (\pi (0.4)^2\, dh)(1 + Ah)(9.8)(h + 0.2)\end{aligned}$$

$$\begin{aligned}\text{Total work} &= \int_0^{0.9} (\pi (0.4)^2)(9.8)(1 + Ah)(h + 0.2)\, dh \\ &= \pi (0.4)^2 (9.8) \int_0^{0.9} (0.2 + h + 0.2Ah + Ah^2)\, dh \\ &= \pi (0.4)^2 (9.8) \left[0.2h + \tfrac{1}{2}h^2 + 0.1Ah^2 + \tfrac{1}{3}Ah^3 \right]_0^{0.9} \\ &= \pi (0.4)^2 (9.8) \left[0.585 + 0.324A \right] \\ &\approx 2.881720109 + 1.596029599\, A \quad \text{N.}\end{aligned}$$

8. A new toy is being developed called the "Infini-bunny." The Infini-bunny jumps up and down on the same spot. To activate the Infini-bunny, a person places the toy on the ground and presses the start button. The Infini-bunny then starts jumping up and down. The first jump is to a height of 10 feet. The second jump is to a height of $10\left(\frac{5}{6}\right)$. The third jump is to a height of $10\left(\frac{5}{6}\right)^2$, and so on. Each jump is $\frac{5}{6}$ of the height of the previous jump. The Infini-bunny will keep jumping up and down until the person catches it and turns it off.

(a) Write down an expression (or formula) for the height (in feet) of the Infini-bunny's n^{th} jump.

$$\text{Height} = 10 \cdot \left(\frac{5}{6}\right)^{n-1}$$

(b) Write down an expression (or formula) for the total vertical distance the Infini-bunny has traveled when it lands at the end of its n^{th} jump. Express your answer in closed form.

NOTE: The old saying, "What goes up must come down" may be useful here and in Part (c).

$$\text{Vertical Distance} = 2(10) + 2 \cdot 10 \cdot \frac{5}{6} + 2 \cdot 10 \cdot \left(\frac{5}{6}\right)^2 + \dots + 2 \cdot 10 \cdot \left(\frac{5}{6}\right)^{n-1}$$

$$= \frac{20 \cdot \left[1 - (5/6)^n\right]}{1 - 5/6}$$

This closed form is possible because the expresion for vertical distance is a finite geometric series with $a = 20$, $r = 5/6$ and a total of n terms.

Continued on the next page.

SOLUTIONS

(c) Find the total vertical distance (in feet) the Infini-bunny will travel if a person activates the Infini-bunny and just leaves it jumping (i.e. they never catch the Infini-bunny and turn it off).

$$\text{Vertical Distance} = 20 + 20\cdot\left(\frac{5}{6}\right) + 20\cdot\left(\frac{5}{6}\right)^2 + \cdots$$

$$= \frac{20}{1 - 5/6}$$

$$= 120 \text{ feet.}$$

This is an infinite geometric series with $a = 20$ and $r = 5/6$.

(d) To complete a jump of height h feet takes the Infini-bunny $\sqrt{\frac{h}{8}}$ seconds. If a person activates the Infini-bunny and then leaves it alone, will the bunny keep jumping forever or does it eventually stop jumping? Briefly justify your answer. Ignore practical considerations like friction, air-resistance, battery life, etc.

$$\text{Time to go up for } n^{th} \text{ jump} = t_n = \sqrt{\frac{10}{8}\left(\frac{5}{6}\right)^{n-1}}$$

$$\text{Total time} = 2\cdot\sqrt{\frac{10}{8}} + 2\sqrt{\frac{10}{8}}\cdot\sqrt{\frac{5}{6}} + 2\sqrt{\frac{10}{8}}\cdot\left(\sqrt{\frac{5}{6}}\right)^2 + \cdots$$

$$= \frac{2\sqrt{10/8}}{1 - \sqrt{5/6}}$$

$$\approx 25.66385658 \text{ seconds.}$$

This is an infinite geometric series with $a = \sqrt{10/8}$ and $r = \sqrt{5/6}$.

SOLUTIONS

9. An object is moving along a curve in the x-y plane. The position of the object at time t is given by the parametric equations $x(t)$ and $y(t)$. All that you can assume about $x(t)$ and $y(t)$ is that their derivatives are given by:

$$\frac{dx}{dt} = t \cdot \cos(t^2 + 1) \quad \text{and} \quad \frac{dy}{dt} = -t \cdot \sin(t^2 + 1)$$

for $0 \le t \le 3$, and that at time $t = 2$ their values are:

- $x(2) = 10$, and,
- $y(2) = -3$.

(a) Write down an equation for the tangent line to the curve at the point $(10, -3)$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-t \cdot \sin(t^2+1)}{t \cdot \cos(t^2+1)} \qquad \left.\frac{dy}{dx}\right|_{t=2} \approx 3.380515006.$$

Equation of Tangent Line:

$$y + 3 = 3.380515006 \cdot (x - 10).$$

(b) Find the speed of the object at time $t = 2$.

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t^2} = |t|.$$

$\boxed{\text{using } \sin^2(t^2+1) + \cos^2(t^2+1) = 1.}$

Plugging in $t = 2$ gives:

$$\text{Speed} = 2.$$

(c) Find the **exact** distance that the object travels in the time interval $0 \le t \le 1$.

$$\text{Exact Distance} = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 t \cdot dt$$

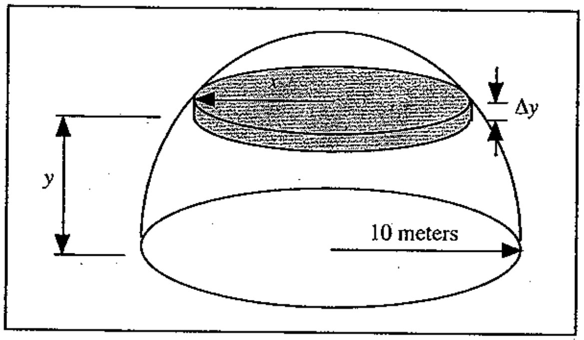
using $\sin^2(t^2+1) + \cos^2(t^2+1) = 1$

$$= \left[\frac{1}{2} t^2\right]_0^1$$

$$= 1/2.$$

SOLUTIONS

10.



Consider the shape shown in the diagram. This shape is the top half of a sphere with a radius of 10 meters. If you look at the shape "side on" then it looks like the top half of the circle described by the equation:

$$x^2 + y^2 = 100.$$

The shape does not have the same density throughout. The density is high near the base of the shape and lower near the top of the shape. In fact, the mass density at a height of y meters is given by the function:

$$p(y) = 2200 - 20y^2 \text{ kilograms per cubic meter.}$$

(a) If you were asked to work out the *mass* of the shape, briefly explain (in a sentence or two why it would be best to slice up the shape into horizontal slices.

We want to slice the shape up into pieces that have approximately constant density throughout. As density depends on y we slice horizontally so that the value of y (hence the density) is approximately constant in each slice.

(b) Set up an integral that will give the *mass* of the shape.

$$\text{Mass} = \int_0^{10} (2200 - 20y^2)\, \pi x^2 \, dy$$

$\boxed{\text{using } x^2 + y^2 = 100}$

$$= \int_0^{10} (2200 - 20y^2)(\pi)(100 - y^2)\, dy$$

(c) Calculate the mass of the object.

$$\text{Mass} = \pi \int_0^{10} (220000 - 4200y^2 + 20y^4)\, dy$$

$$= \pi \left[220\,000\, y - \frac{4200}{3} y^3 + \frac{20}{5} y^5 \right]_0^{10}$$

$$= \pi \cdot 1200000$$

$$\approx 3769911.184 \text{ Kg}$$

SOLUTIONS

11. Find solutions to the differential equations below, subject to the given initial conditions. In each case, provide step-by-step work to show how your answer was obtained. If you do not provide any work to justify your answer, you will get zero credit.

(a) $\dfrac{dP}{dt} = 3P - 6$ $\qquad P(0) = 20.$

Use Separation of Variables: $\dfrac{dP}{dt} = 3(P-2)$

$$\int \frac{1}{P-2}\, dP = \int 3\, dt$$

$$\ln(|P-2|) = 3t + C$$

$$P - 2 = A e^{3t}, \quad A = \pm e^{C}$$

To determine A, use $P(0) = 20$, giving $A = 18$.

Final answer: $P(t) = 2 + 18 \cdot e^{3t}$

(b) $\dfrac{dy}{dx} = \dfrac{5y}{x}$ $\qquad y(2) = 64.$

Use Separation of Variables:

$$\int \frac{1}{y}\, dy = \int \frac{5}{x}\, dx$$

$$\ln(|y|) = 5 \cdot \ln(|x|) + C$$

$$y = A \cdot x^5, \quad A = \pm e^{c}$$

To determine A, use $y(2) = 64$ to get $A = 2$.

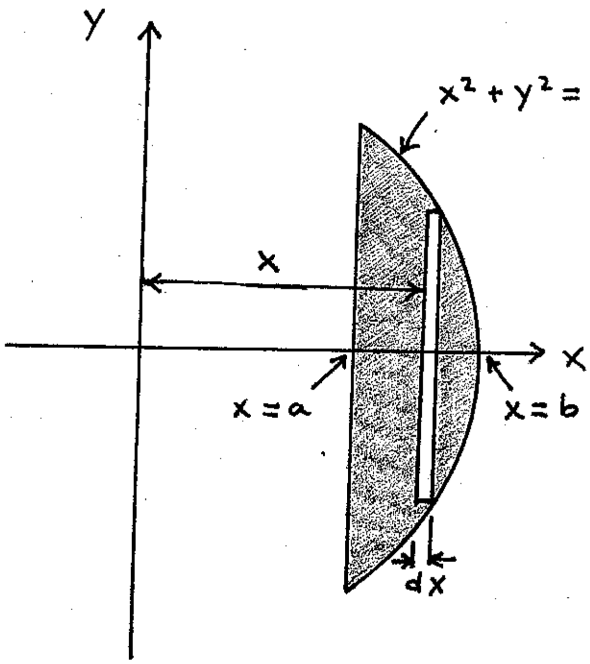
Final answer: $y = 2 \cdot x^5$.

SOLUTIONS

12. A perfectly spherical pearl has a radius of b mm, where b is a positive number. To prepare it for sale, a circular hole is bored through the center of the pearl. The radius of the hole is a mm, where a is a positive number and $0 < a < b$. The circular hole goes completely through the pearl, from one side of the pearl to the opposite side of the pearl.

Before the circular hole was bored, the volume of the pearl was $\frac{4}{3}\pi b^3$. Find the volume of the pearl that remains after the hole has been bored. Remember to include appropriate units with your answer.

We will calculate the volume of the pearl after the hole was bored by revolving the shaded region shown below around the y-axis. We will use Shells to do this calculation.



Shell volume

$$= 2\pi x \cdot (2y) \cdot dx$$

$$= 4\pi x \cdot \sqrt{b^2 - x^2}\, dx$$

Total volume

$$= \int_a^b 4\pi x \cdot \sqrt{b^2 - x^2} \cdot dx$$

$$= \left[-2\pi \cdot \frac{2}{3} \cdot (b^2 - x^2)^{3/2} \right]_a^b$$

$$= \frac{4\pi}{3} (b^2 - a^2)^{3/2}$$

units: cubic millimeters.

SOLUTIONS

13. Suppose that $g(0) = 2$, $g(3) = 5$ and $\int_0^3 g(x)dx = 7$. Calculate the exact numerical value of each of the following definite integrals.

(i) $\int_0^3 g(3-x)dx = -\int_3^0 g(u)\,du = \int_0^3 g(u)\,du = 7.$

$u = 3 - x$

$dx = -du$

u-substitution.

x	0	3
u	3	0

(ii) $\int_0^9 g\left(\frac{x}{3}\right)dx = 3\int_0^3 g(u)\,du = 21.$

$u = x/3$

$dx = 3\,du$

u-substitution.

x	0	9
u	0	3

(iii) $\int_0^3 x \cdot g'(x)dx = x \cdot g(x)\Big|_0^3 - \int_0^3 g(x)\,dx$

$u = x \quad v' = g'$

$u' = 1 \quad v = g$

$= (3)(5) - (0)(2) - 7$

$= 8$

Integration by Parts.

14. Suppose that f is a twice differentiable function with $f(0) = 6$, $f(1) = 5$ and $f'(1) = 2$. Calculate the exact value of $\int_0^1 x \cdot f''(x)dx$. Show your work.

Integrate by parts with $u = x$ and $v' = f''(x)$. Then $u' = 1$ and $v = f'(x)$, so:

$$\int_0^1 x \cdot f''(x) \cdot dx = \Big[x \cdot f'(x)\Big]_0^1 - \int_0^1 f'(x)\,dx$$

$$= (1)(2) - (0)\cdot f'(0) - \Big[f(x)\Big]_0^1$$

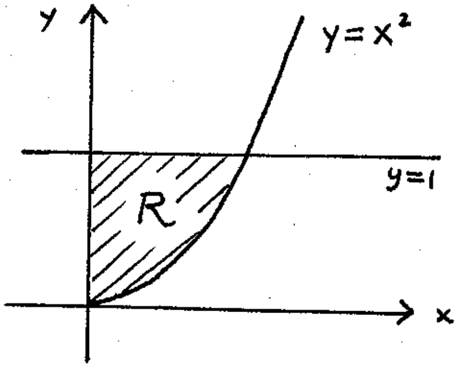
$$= 2 - f(1) + f(0)$$

$$= 2 - 5 + 6$$

$$= 3$$

15. Use integrals to calculate the volume described in each part of this problem.

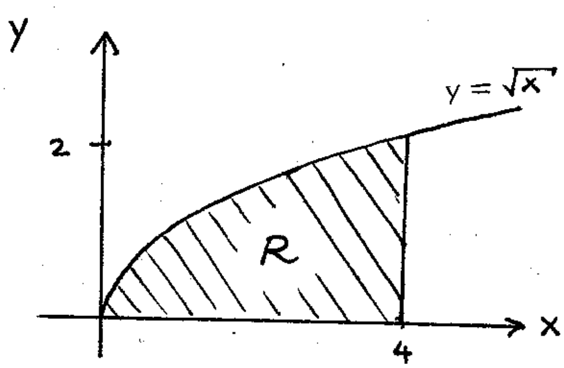
(a) The region R is bounded by $y = x^2$, $y = 1$ and the y-axis. Find the volume of the solid obtained by rotating the region R around the y-axis.



We can slice the area up into horizontal rectangles and use the method of disks.

$$\text{Volume} = \int_0^1 \pi x^2 \, dy$$
$$= \int_0^1 \pi y \, dy$$
$$= \left[\tfrac{1}{2} \pi y^2 \right]_0^1$$
$$= \tfrac{1}{2} \pi$$

(b) A region R of the xy-plane is enclosed by the curves: $y = 0, x = 0,\ y = \sqrt{x}$ and $x = 4$. Find the volume generated when the region R is revolved around the y-axis.



You could find the volume using shells or washers. We will set this up using shells.

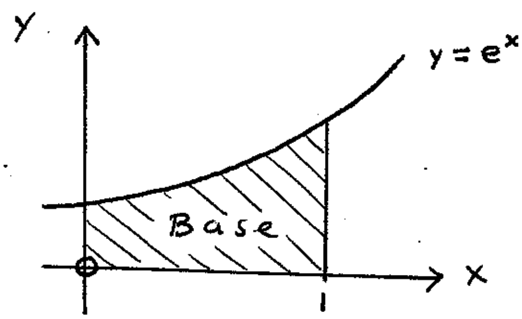
$$\text{Volume} = \int_0^4 2\pi x \cdot \sqrt{x} \cdot dx$$
$$= \int_0^4 2\pi \cdot x^{3/2} \cdot dx$$
$$= \left[2\pi \cdot \frac{2}{5} x^{5/2} \right]_0^4$$
$$= \frac{128\pi}{5}$$

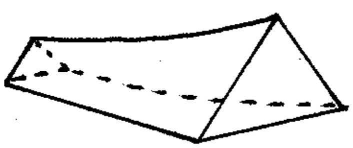
Continued on the next page

(c) Calculate the volume of the solid whose base is the region bounded by:

- $y = e^x$;
- the x-axis, and,
- the lines $x = 0$ and $x = 1$

and whose cross-sections are equilateral triangles that are perpendicular to the x-axis.



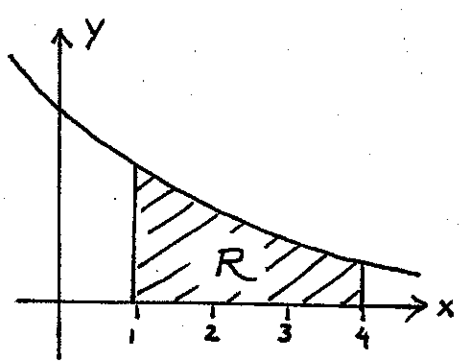


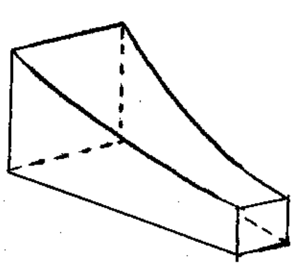
$$\begin{aligned} \text{Volume} &= \int_0^1 \frac{\sqrt{3}}{4} \cdot y^2 \, dx \\ &= \int_0^1 \frac{\sqrt{3}}{4} (e^x)^2 \, dx \\ &= \int_0^1 \frac{\sqrt{3}}{4} e^{2x} \, dx \\ &= \left[\frac{\sqrt{3}}{8} e^{2x} \right]_0^1 \\ &= \frac{\sqrt{3}}{8} (e^2 - 1). \end{aligned}$$

(d) The region R is bounded by:

- The line $y = 0$,
- The line $x = 1$,
- The line $x = 4$, and by
- The curve $y = e^{-x}$.

Consider the solid whose base is the region R, and whose cross-sections perpendicular to the x-axis are squares. Calculate the **exact** volume of this solid.





$$\begin{aligned} \text{Volume} &= \int_1^4 y^2 \, dx \\ &= \int_1^4 (e^{-x})^2 \, dx \\ &= \int_1^4 e^{-2x} \, dx \\ &= \left[\frac{e^{-2x}}{-2} \right]_1^4 \\ &= \frac{1}{2} (e^{-2} - e^{-8}) \end{aligned}$$

SOLUTIONS

16. (a) Using the techniques of integration that you have learned (and not calculator integration), find the *exact* value of the definite integral:

$$\int_{-b}^{b}\left(\frac{1}{1+x^2}+\frac{x}{1+x^2}\right)dx.$$

Your final answer may contain the symbol b but should contain no other unspecified constants.

Note that: $\displaystyle\int\frac{1}{a^2+x^2}\,dx=\frac{1}{a}\arctan\left(\frac{x}{a}\right)+C.$

For $\displaystyle\int_{-b}^{b}\frac{1}{1+x^2}\,dx$ we can use the given integration formula to obtain:

$$\int_{-b}^{b}\frac{1}{1+x^2}\,dx = \Big[\arctan(x)\Big]_{-b}^{b}$$

$$= 2\cdot\arctan(b)$$

For $\displaystyle\int_{-b}^{b}\frac{x}{1+x^2}\,dx$ we can carry out a u-substitution with $u=1+x^2$ or notice that since $f(x)=\dfrac{x}{1+x^2}$ is an odd function, $\displaystyle\int_{-b}^{b}\frac{x}{1+x^2}\,dx = 0.$

Adding gives:

$$\int_{-b}^{b}\left(\frac{1}{1+x^2}+\frac{x}{1+x^2}\right)dx = 2\cdot\arctan(b).$$

Continued on the next page.

SOLUTIONS

(b) Use your answer to Part (a) to calculate the limit: $\lim_{b \to \infty} \int_{-b}^{b} \left(\frac{1}{1+x^2} + \frac{x}{1+x^2} \right) dx$.

Note that $\text{Lim}_{x \to \infty} \arctan(x) = \frac{\pi}{2}$.

$$\lim_{b \to \infty} \int_{-b}^{b} \left(\frac{1}{1+x^2} + \frac{x}{1+x^2} \right) dx = \lim_{b \to \infty} 2 \cdot \arctan(b)$$

$$= 2 \cdot \pi/2$$

$$= \pi.$$

(c) Does the improper integral $\int_{-\infty}^{\infty} \frac{1+x}{1+x^2} dx$ converge or diverge? Briefly explain how you know.

The improper integral $\int_{-\infty}^{\infty} \frac{1+x}{1+x^2} dx$ diverges.

This is because in order for this integral to converge, both of the integrals $\int_{-\infty}^{0} \frac{1+x}{1+x^2} dx$ and $\int_{0}^{\infty} \frac{1+x}{1+x^2} dx$ have to converge independently of each other.

$$\int_0^\infty \frac{1+x}{1+x^2} dx = \lim_{a \to \infty} \int_0^a \frac{1}{1+x^2} + \frac{x}{1+x^2} dx$$

$$= \lim_{a \to \infty} \Big[\arctan(x) + \tfrac{1}{2} \ln(|1+x^2|) \Big]_0^a$$

$$= \lim_{a \to \infty} \arctan(a) + \tfrac{1}{2} \ln(1+a^2)$$

$$= +\infty$$

Since this improper integral diverges, $\int_{-\infty}^{\infty} \frac{1+x}{1+x^2} dx$ also diverges.

SOLUTIONS

17. On afternoon in January, my furnace broke at 1:00pm. At the time this happened, the temperature in the house was 68°F. When I got home at 7:00pm, the temperature in the house was 48°F while the outside temperature was 10°F. You can assume that this was an unusual day during which the outside temperature stayed at exactly 10°F all day.

(a) Assuming that the temperature T in my house obeys Newton's Law of Cooling, write down the differential equation for T. (This will involve one unknown constant that you will have to find later.)

Newton's Law of Cooling: $\frac{dT}{dt} = k \cdot [T - 10]$ t = hours since 1 pm

k = unknown constant.

Function values: $T(0) = 68$ $T(6) = 48$

(b) Solve the differential equation from Part (a).

Use Separation of Variables: $\int \frac{1}{T-10}\,dT = \int k\,dt$

$$\ln(|T-10|) = kt + C$$

$$T - 10 = Ae^{kt}, \quad A = \pm e^{c}$$

To determine A, use $T(0) = 68$: $68 - 10 = Ae^{k(0)}$

$$58 = A.$$

To determine k, use $T(6) = 48$: $48 - 10 = 58 \cdot e^{6k}$

$$k = \frac{1}{6}\ln\left(\frac{38}{58}\right) \doteq -0.0704761418$$

Final Answer: $T(t) = 10 + 58e^{-0.0704761418t}$

(c) Assuming that the furnace is not repaired, at what time will the temperature in the house reach the freezing point of 32°F?

Set $T = 32$ and solve for t:

$$32 = 10 + 58e^{-0.0704761418t}$$

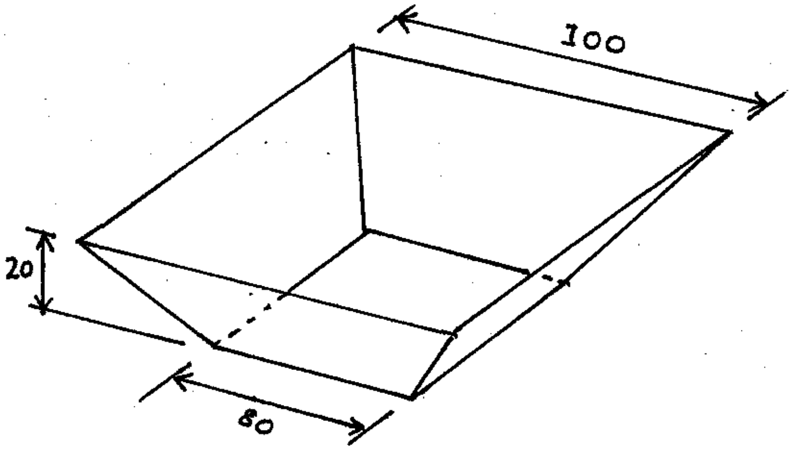
$$t = 13.7550174l$$

This corresponds to about 2:45 am.

SOLUTIONS

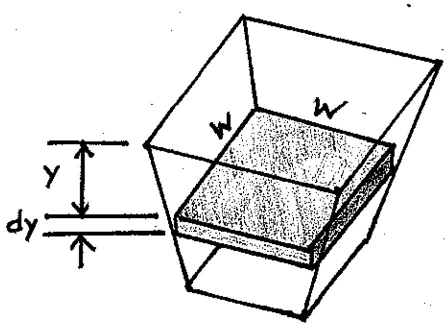
18. On Staten Island, garbage is dumped into a pyramid-shaped hole with a square base. (The base is at ground level; the hole gets narrower as you go down into the earth.) The length of each side of the square base is 100 yards. For each one yard you go down vertically in the hole, the length of the sides decreases by a yard. For example, if you go down one yard vertically, then the length of each side of the pyramid-shaped hole is 99 yards. Initially the hole is 20 feet deep.

(a) If 65 cubic yards of garbage arrive at the dump each day, how long will it be (in days) before the dump is full?



Note that the hole is not a perfect pyramid. "Frustrum pyramid" is probably a more accurate description.

First we must calculate the volume of the hole. To do this we will slice the hole horizontally.



Volume of shaded slice $= W^2 \cdot dy$

$= (100-y)^2 \cdot dy$

Total volume $= \int_0^{20} (100-y)^2\, dy$

$= \left[-\dfrac{(100-y)^3}{3} \right]_0^{20}$

$= \dfrac{100^3 - 80^3}{3}$ cubic yards

Continued on the next page.

At 65 cubic yards per day, it will take $\dfrac{100^3 - 80^3}{195} \approx 2503$ days to fill the hole.

SOLUTIONS

(b) Suppose garbage weighs 800 pounds per cubic yard. A few years after the hole is completely filled with garbage, environmentalists force the city to dig up all the garbage and remove it from the hole. How much work must be done to remove the garbage from the hole?

Work done to remove shaded slice $= (\text{Force})(\text{Distance})$

$$= (800)(100-y)^2 \cdot dy \cdot (y)$$

Total work $= \int_0^{20} (800)\, y\, (100-y)^2 \cdot dy$

$$= 800 \int_0^{20} (10\,000\, y - 200 y^2 + y^3)\, dy$$

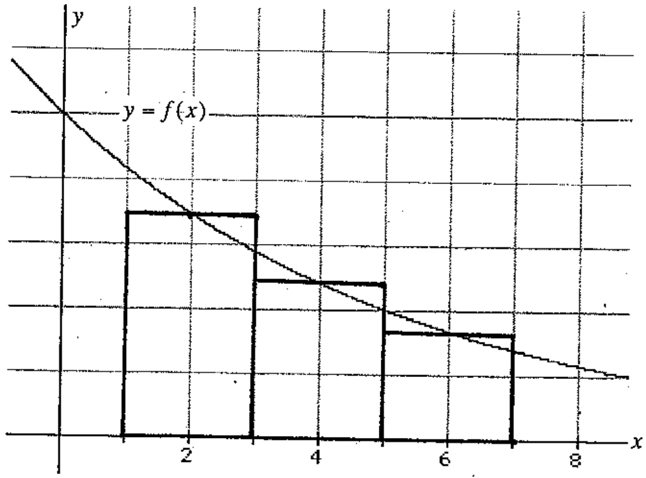
$$= 800 \left[5000\, y^2 - \frac{200}{3} y^3 + \frac{1}{4} y^4 \right]_0^{20}$$

$$= 120533333 \text{ yard - pounds}$$

$$= 361600000 \text{ foot - pounds.}$$

SOLUTIONS

19. The graph of a function $f(x)$ is shown below.



(a) Use the graph given above to sketch the rectangles that you would use if you were asked to estimate the value of the definite integral $\int_1^7 f(x)dx$ using the *midpoint rule* and a total of three (3) rectangles.

See diagram above.

(b) The table (below) gives selected values of $f(x)$. Use the *trapezoid rule*, the values given in the table and a total of three (3) rectangles to approximate the value of the definite integral $\int_1^7 f(x)dx$.

Left hand sum $= 2(4.2 + 2.9 + 2.0) = 18.2$

Right hand Sum $= 2(2.9 + 2.0 + 1.4) = 12.6$

x	1	2	3	4	5	6	7
$f(x)$	4.2	3.5	2.9	2.4	2.0	1.7	1.4

Trapezoid Rule $= \dfrac{18.2 + 12.6}{2} = 15.4$

(c) Is the value that you calculated in Part (b) an over- or an under-estimate of the definite integral? Briefly explain how you can tell.

It is an overestimate of $\int_1^7 f(x)dx$. This is because $f(x)$ is concave up and the Trapezoid rule over-estimates the area beneath a concave up curve.

(d) Give an example of a function for which the trapezoid and midpoint rules will give exactly the same value.

Any linear or constant function will work.

e.g. $f(x) = x$.

SOLUTIONS

20. (a) Find the approximate value of $\int_1^4 \sin\left(\sqrt{x}\right)dx$ obtained when the integral is approximated using the Midpoint Rule and 25 rectangles.

1 $\boxed{\text{STO}\triangleright}$ A   4 $\boxed{\text{STO}\triangleright}$ B   25 $\boxed{\text{STO}\triangleright}$ N   (B − A)/N $\boxed{\text{STO}\triangleright}$ W

Y1 = sin(√(X))

sum(seq(Y1(A + 0.5*W + K*W)*W, K, 0, N−1)) = 2.881069247

(b) Find the approximate value of $\int_1^4 \sin\left(\sqrt{x}\right)dx$ obtained when the integral is approximated using the Trapezoid Rule and 25 trapezoids.

Left hand sum = 2.876326353   Right hand sum = 2.884465526

$$\text{Trapezoid Rule} = \frac{2.876326353 + 2.884465526}{2} = 2.880395939$$

(c) Find the approximate value of $\int_1^4 \sin\left(\sqrt{x}\right)dx$ obtained when the integral is approximated using Simpson's Rule and 50 rectangles.

$$\text{Simpson} = \frac{1}{3}(\text{Trapezoid}) + \frac{2}{3}(\text{Midpoint})$$

$$= 2.880844811$$

(d) How many rectangles should you use if you wanted to approximate the value of $\int_1^4 \sin\left(\sqrt{x}\right)dx$ using the Midpoint Rule and with an error of less than 0.005? Show your work.

We want to find N so that: $\dfrac{K\cdot(4-1)^3}{24\cdot N^2} < 0.005.$

K is the maximum value of $|f''(x)|$ over the interval $[1,4]$.

$f(x) = \sin(\sqrt{x})$ $f'(x) = \cos(\sqrt{x})\cdot\frac{1}{2}x^{-1/2}$

$f''(x) = -\sin(\sqrt{x})\cdot\frac{1}{4}\cdot x^{-1} - \cos(\sqrt{x})\cdot\frac{1}{4}x^{-3/2}.$

Graphing $|f''(x)|$ over the interval $[1,4]$ gives:

$K = 0.34544332.$

To find N: $\dfrac{(0.34544332)(3)^3}{24\cdot N^2} < 0.005.$

$$N > \sqrt{\frac{(0.34544332)(3)^3}{(24)(0.005)}} \approx 8.816$$

Use at least 9 rectangles.

SOLUTIONS

21. Use the technique of Trigonometric substitution to evaluate the definite integrals shown below.

(a) $$\int_1^2 \frac{1}{y^2 \cdot \sqrt{5-y^2}} \cdot dy$$

Use trigonometric substitution with $y = \sqrt{5} \cdot \cos(\theta)$.

Then : $$5 - y^2 = 5 - 5\cos^2(\theta) = 5 \cdot \sin^2(\theta)$$

$$\sqrt{5-y^2} = \sqrt{5} \cdot \sin(\theta).$$

$$\frac{dy}{d\theta} = -\sqrt{5}\,\sin(\theta) \quad \text{so} \quad dy = -\sqrt{5}\,\sin(\theta)\,d\theta.$$

Substituting :

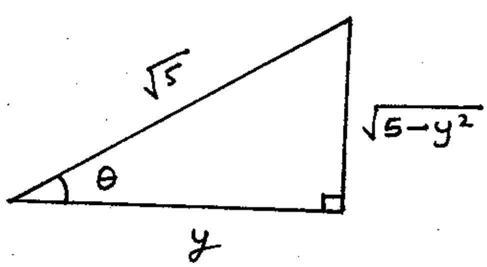
$$\int \frac{1}{y^2\sqrt{5-y^2}}\,dy = \int \frac{-\sqrt{5} \cdot \sin(\theta)}{5 \cdot \cos^2(\theta) \cdot \sqrt{5}\,\sin(\theta)}\,d\theta$$

$$= \frac{-1}{5}\int \sec^2(\theta)\,d\theta$$

$$= \frac{-1}{5}\tan(\theta) + C$$

Next, convert the antiderivative back to y using a right-angle triangle. Using the triangle shown below, $\tan(\theta) = \dfrac{\sqrt{5-y^2}}{y}$ so :

$$\frac{y}{\sqrt{5}} = \cos(\theta)$$



$$\int_1^2 \frac{1}{y^2\sqrt{5-y^2}}\,dy = \left[\frac{\sqrt{5-y^2}}{-5y}\right]_1^2$$

$$= \frac{-1}{10} - \frac{-2}{5}$$

$$= \frac{3}{10}.$$

Continued on the next page.

SOLUTIONS

(b) $$\int_0^{\pi/2} \frac{\cos(z)}{\sqrt{1+\sin^2(z)}}\,dz$$

HINT: (a) Try a u-substitution to begin with; $u = 1+\sin^2(z)$ is probably not going to be very helpful.

(b) You may use the integration formula: $\int \sec(x)dx = \ln(|\sec(x)+\tan(x)|)+C$.

Let $u = \sin(z)$ so that $dz = \frac{1}{\cos(z)}\,du$ and:

$$\int \frac{\cos(z)}{\sqrt{1+\sin^2(z)}}\,dz = \int \frac{1}{\sqrt{1+u^2}}\,du.$$

Next make the trigonometric substitution $u = \tan(\theta)$ so that $\sqrt{1+u^2} = \sec(\theta)$ and $du = \sec^2(\theta)\,d\theta$:

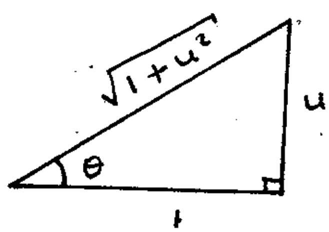
$$\int \frac{1}{\sqrt{1+u^2}}\,du = \int \frac{1}{\sec(\theta)}\sec^2(\theta)\,d\theta$$

$$= \int \sec(\theta)\,d\theta$$

$$= \ln(|\sec(\theta)+\tan(\theta)|)+C$$

To convert the antiderivative back to u, use the right-angle triangle shown below. $\sec(\theta) = \sqrt{1+u^2}$ so:

$\frac{u}{1} = \tan(\theta)$



$$\int \frac{1}{\sqrt{1+u^2}}\,du = \ln\left(\left|\sqrt{1+u^2}+u\right|\right)+C$$

and then finally, convert back to z using $u = \sin(z)$.

$$\int_0^{\pi/2} \frac{\cos(z)}{\sqrt{1+\sin^2(z)}}\,dz = \left[\ln\left(\left|\sqrt{1+\sin^2(z)}+\sin(z)\right|\right)\right]_0^{\pi/2} = \ln(\sqrt{2}+1).$$

SOLUTIONS

22. In this problem, all that you may assume that $f(x)$ is an **odd** function, that $g(x)$ is an **even** function, that the domains of both functions include $-4 \le x \le 4$ and that they have the values shown below.

$f(-2) = 5$ $\qquad$ $f(0) = 0$ $\qquad$ $g(2) = -7$ $\qquad$ $g'(2) = -1.$

(a) Find the exact value of the definite integral: $\int_{-3}^{3} f(x)^3 \cdot [1 + g(x)] \cdot dx$.

$f(x)^3$ is odd and $1 + g(x)$ is even so the product is odd and $\int_{-3}^{3} f(x)^3 \cdot [1 + g(x)] \cdot dx = 0.$

(b) Find the exact value of the definite integral: $\int_{-2}^{2} x \cdot f'(x) dx$.

Integrate by Parts with $u = x$ and $v' = f'$:

$$\int_{-2}^{2} x \cdot f'(x)\, dx = \Big[x \cdot f(x)\Big]_{-2}^{2} - \int_{-2}^{2} f(x)\, dx$$

$$= (2)\cdot(-5) - (-2)(5) - 0 = 0.$$

(c) Find the exact value of the definite integral: $\int_{0}^{2} f'(x) \cdot \sqrt{10 + f(x)} \cdot dx$.

Use u-substitution with $u = 10 + f(x)$.

$$\int_0^2 f'(x)\sqrt{10 + f(x)}\, dx = \int_{10}^{5} u^{1/2}\, du = \Big[\tfrac{2}{3} u^{3/2}\Big]_{10}^{5}$$

$$= \tfrac{2}{3}\left(5^{3/2} - 10^{3/2}\right)$$

(d) Find the exact value of the definite integral: $\int_{0}^{\sqrt{2}} x \cdot f'(x^2) dx$.

Use u-substitution with $u = x^2$.

$$\int_0^{\sqrt{2}} x \cdot f'(x^2) \cdot dx = \int_0^2 \tfrac{1}{2} f'(u)\, du = \Big[\tfrac{1}{2} f(u)\Big]_0^2$$

$$= -\frac{5}{2}$$

23. (a) Find a formula for the most general antiderivative of: $f(x)=\dfrac{e^x}{(e^x-M)(e^x+3N)}$ where M and N are constants and $3N+M\neq 0$.

We will begin with the u-substitution $u=e^x$. Doing this:

$$\int f(x)\,dx = \int \frac{1}{(u-M)(u+3N)}\,du.$$

Now we will use partial fractions on the transformed integral.

$$\frac{1}{(u-M)(u+3N)} = \frac{A}{u-M} + \frac{B}{u+3N}$$

Finding the constants A and B:

$$A\cdot(u+3N) + B\cdot(u-M) = 1$$

Coefficient of u: $A + B = 0$

Constants: $3N\cdot A - MB = 1$

So, $B=-A$ and $3N\cdot A + M\cdot A = 1$ giving:

$$A = \frac{1}{3N+M} \qquad B = \frac{-1}{3N+M}.$$

Therefore:

$$\int f(x)\,dx = \int \frac{\frac{1}{3N+M}}{u-M}\,du - \int \frac{\frac{1}{3N+M}}{u+3N}\,du$$

$$= \frac{1}{3N+M}\ln(|u-M|) - \frac{1}{3N+M}\ln(|u+3N|)+C$$

$$= \frac{1}{3N+M}\ln(|e^x-M|) - \frac{1}{3N+M}\ln(|e^x+3N|)+C$$

Continued on the next page.

SOLUTIONS

(b) Find a formula for the most general antiderivative of: $g(x) = \dfrac{1}{x^2 - 6Bx + 10B^2}$ where B is a positive constant.

We will use the integration formula:

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a} \cdot \arctan\left(\frac{x}{a}\right) + C.$$

To put $g(x)$ into a format compatible with this integration formula, we will complete the square in the denominator.

$$x^2 - 6Bx + 10B^2 = x^2 - 6Bx + 9B^2 - 9B^2 + 10B^2$$

$$= (x - 3B)^2 + B^2.$$

Therefore:

$$\int g(x)\,dx = \int \frac{1}{B^2 + (x - 3B)^2}\,dx$$

$$= \frac{1}{B} \cdot \arctan\left(\frac{x - 3B}{B}\right) + C$$

Continued on the next page.

SOLUTIONS

(c) Find a formula for the most general antiderivative of: $j(x) = \dfrac{x^3 + Rx^2 - R^2x - R^3}{x - R}$ where R is a positive constant.

We will use polynomial long division to simplify $j(x)$ before finding the antiderivative.

$$
\begin{array}{r|l}
 & x^2 + 2Rx + R^2 \\
\hline
x - R & x^3 + Rx^2 - R^2x - R^3 \\
 & \underline{x^3 - Rx^2} \\
 & \quad 2Rx^2 - R^2x \\
 & \quad \underline{2Rx^2 - 2R^2x} \\
 & \qquad\qquad R^2x - R^3 \\
 & \qquad\qquad \underline{R^2x - R^3} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

Based on this quotient,

$$\int j(x)\,dx = \int \left(x^2 + 2Rx + R^2\right) dx$$

$$= \frac{1}{3}x^3 + Rx^2 + R^2x + C$$

SOLUTIONS

24. Evaluate each of the indefinite integrals to find the most general antiderivative.

(a) $\int \cos^4(x)\cdot\tan^3(x)\cdot dx$

Note that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ so that:

$$\cos^4(x)\cdot\tan^3(x) = \cos(x)\cdot\sin^3(x).$$

Then, carrying out a u-substitution with $u=\sin(x)$ gives:

$$\int \cos^4(x)\tan^3(x)\,dx = \int \cos(x)\cdot\sin^3(x)\,dx$$
$$= \int u^3\,du$$
$$= \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4(x) + C.$$

(b) $\int \sec^6(\theta)\cdot d\theta$

Note that $\tan^2(\theta)+1 = \sec^2(\theta)$ so that:

$$\sec^4(\theta) = (\tan^2(\theta)+1)^2 = \tan^4(\theta) + 2\cdot\tan^2(\theta) + 1.$$

Therefore:

$$\int \sec^6(\theta)\,d\theta = \int \sec^4(\theta)\cdot\sec^2(\theta)\,d\theta$$
$$= \int (\tan^4(\theta) + 2\tan^2(\theta) + 1)\cdot\sec^2(\theta)\,d\theta$$
$$= \int \tan^4(\theta)\cdot\sec^2(\theta)\,d\theta + 2\int \tan^2(\theta)\sec^2(\theta)\,d\theta + \int \sec^2(\theta)\,d\theta$$
$$= \frac{1}{5}\tan^5(\theta) + \frac{2}{3}\tan^3(\theta) + \tan(\theta) + C.$$

SOLUTIONS

25. Find the solution to the following initial value problem. Note that your final answer should not contain any unspecified constants. Clearly indicate your final answer.

$$y'' + 16 \cdot y = \sin(x) \qquad y(0) = 1 \qquad y'(0) = 0$$

Homogeneous Solution:

$$\left.\begin{aligned} y'' + 16y &= 0 \\ r^2 + 4y &= 0 \end{aligned}\right\} \text{ Roots } r = \pm 4i$$

$$y_h(x) = C_1 \cos(4x) + C_2 \sin(4x).$$

Particular Solution:

$$\left.\begin{aligned} N(x) &= \sin(x) \\ N'(x) &= \cos(x) \\ N''(x) &= -\sin(x) \end{aligned}\right\} \text{ What's important: } \sin(x),\ \cos(x)$$

(Repeating so stop.)

$$y_p(x) = F \cdot \cos(x) + G \cdot \sin(x)$$

Determine Constants F, G:

$$y_p'(x) = -F \sin(x) + G \cos(x)$$

$$y_p''(x) = -F \cos(x) - G \sin(x).$$

$$y_p''(x) + 16\, y_p(x) = \sin(x)$$

$$-F\cos(x) - G\sin(x) + 16F\cos(x) + 16G\sin(x) = \sin(x).$$

Coefficient of cos(x): $15F = 0$ so $F = 0$

Coefficient of sin(x): $15G = 1$ so $G = 1/15$

$$y_p(x) = 1/15 \sin(x)$$

Determine Constants C_1, C_2:

$$y(x) = y_h(x) + y_p(x)$$

$$= C_1 \cos(4x) + C_2 \sin(4x) + 1/15 \sin(x)$$

$y(0) = 1$: $C_1 = 1$ so $C_1 = 1$

$y'(0) = 0$: $4C_2 + 1/15 = 0$ so $C_2 = -1/60$

Final Answer: $y(x) = \cos(4x) - \frac{1}{60}\sin(4x) + \frac{1}{15}\sin(x)$

SOLUTIONS

26. The function $y = f(x)$ is defined by the following differential equation and initial condition:

Differential equation: $f'(x) = x^2 + f(x)$ or $y' = x^2 + y$

Initial value: $f(0) = 1$ or $y(0) = 1$.

(a) Use Euler's Method and $\Delta x = 0.25$ to estimate $f(2)$. Clearly indicate your final answer.

Current t	Current f(x)	f'(x)	Rise = f'(x) Δx	New f(x)
0	1	1	0.25	1.25
0.25	1.25	1.3125	0.328125	1.578125
0.5	1.578125	1.828125	0.45703125	2.03515625
0.75	2.03515625	2.59765625	0.6494140625	2.684570313
1.0	2.684570313	3.684570313	0.921142578	3.605712891
1.25	3.605712891	5.168212891	1.292053223	4.897766113
1.5	4.897766113	7.147766113	1.786941528	6.684707642
1.75	6.684707642	9.747207642	2.43680191	9.121509552
2.0	9.121509552			

$f(2) \approx 9.121509552.$

(b) Is the estimate of $f(2)$ that you calculated in Part (a) an over-estimate or an under-estimate of the actual value of $f(2)$? Be careful to show your work and explain how you know.

The estimate $f(2) \approx 9.121509552$ is an underestimate of the correct value of $f(2)$.

This is because the increasing values of $f'(x)$ (see middle column above) show that $f(x)$ is concave up. When $f(x)$ is concave up, Euler's method underestimates the function values.

SOLUTIONS

27. A tank initially contains 120 liters of pure water. A saline solution (with a concentration of γ grams per liter of salt) is pumped into the tank at a rate of 2 liters per minute. The well-stirred mixture leaves the tank at the same rate. In this problem, t is the time (in minutes) after the saline solution started pumping.

(a) Write down an initial value problem (i.e. a differential equation and a function value) for $u(t)$, the mass of salt in the tank at time t. Assume $u(t)$ is in units of grams.

Rate in $= (2)(\gamma)$ Rate out $= (2) \cdot \dfrac{u(t)}{120}$

Differential Equation: $$\frac{du}{dt} = 2\gamma - \frac{1}{60} u(t)$$

Initial Condition: $u(0) = 0.$

(b) Find an explicit formula for $u(t)$.

Solve Differential Equation: $$\frac{du}{dt} = \frac{-1}{60}\left(u - 120\gamma \right)$$

$$\int \frac{1}{u - 120\gamma}\, du = \int \frac{-1}{60}\, dt$$

$$\ln(|u - 120\gamma|) = \frac{-1}{60} t + C$$

$$u - 120\gamma = Ae^{-\frac{1}{60}t}, \quad A = \pm e^{C}$$

Evaluate Constant A: $$-120\gamma = A \quad (\text{use } u(0) = 0).$$

Final Answer: $$u(t) = 120\gamma - 120\gamma e^{-\frac{1}{60}t}$$

(c) Find an expression for the limit of $u(t)$ as $t \to \infty$.

$$\lim_{t \to \infty} u(t) = \lim_{t \to \infty} 120\gamma - 120\gamma e^{-\frac{1}{60}t} = 120\gamma.$$ grams of salt.

SOLUTIONS

28. In this problem you will be concerned with the differential equation: $y'' - 3y' - 4y = e^t$.

(a) Find the general solution of the homogeneous equation:

$$y'' - 3y' - 4y = 0.$$

Characteristic Equation: $r^2 - 3r - 4 = 0$, $(r-4)(r+1) = 0$; Roots: $r = 4,\ r = -1$

Homogeneous Solution: $y_h(t) = C_1 e^{4t} + C_2 e^{-t}$

(b) Find a particular solution of the non-homogeneous equation:

$$y'' - 3y' - 4y = e^t.$$

Here $N(t) = e^t$. $N'(t) = e^t$ so we have repetition immediately.

Particular Solution: $y_p(t) = Fe^t$. (F = constant)

Determine value of F:

$$y_p''(t) - 3y_p'(t) - 4y_p(t) = e^t$$
$$Fe^t - 3Fe^t - 4Fe^t = e^t$$
$$-6Fe^t = e^t$$
$$\therefore F = -1/6$$

Particular Solution: $y_p(t) = -\frac{1}{6} e^t$

(c) Find a solution of the initial value problem:

$$y'' - 3y' - 4y = e^t \qquad y(0) = 1 \qquad y'(0) = 0.$$

Solution of DE:

$$y(t) = y_h(t) + y_p(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{6} e^t$$

Determine Constants:

$y(0) = 1$: $c_1 + c_2 - 1/6 = 1$

$y'(0) = 0$: $4c_1 - c_2 - 1/6 = 0$

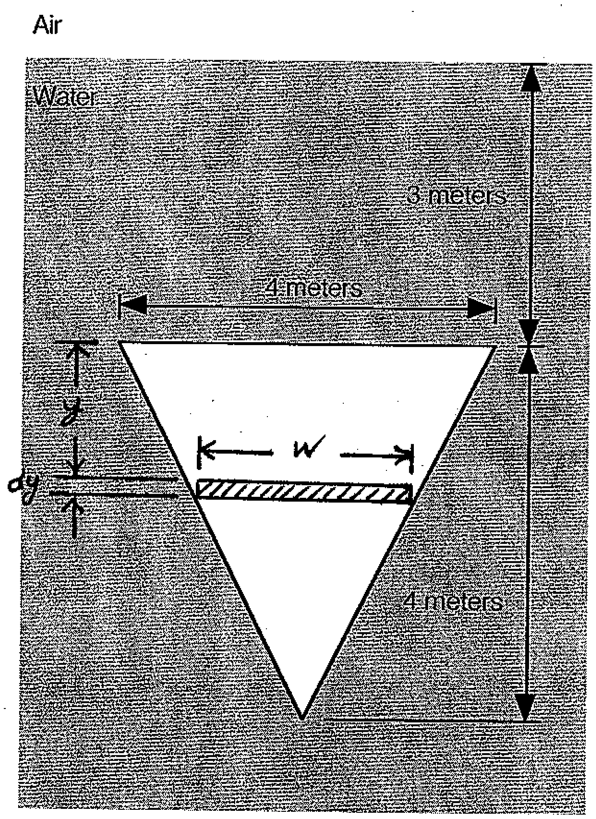
Adding gives: $5c_1 = 1/3$ so $c_1 = 1/15$.

Then: $1/15 + c_2 - 1/6 = 1$ so $c_2 = 33/30 = 11/10$

Final Answer: $y(t) = \frac{1}{15} e^{4t} + \frac{11}{10} e^{-t} - \frac{1}{6} e^t$

SOLUTIONS

29. A triangular plate is submerged in a tank of fresh water (density = 1000 kg/m^3) as shown in the diagram below. The top of the triangular plate is three (3) meters below the surface of the water and the triangular plate is vertical in the water. Calculate the hydrostatic force exerted by the water on one side of the triangular plate.



Let y = distance (vertical) down from the top of the plate.

The force on the shaded rectangle shown is:

$$\text{Force} = \underbrace{(1000)(9.8)(3+y)}_{\text{pressure} = \rho \cdot g \cdot h}\ \underbrace{w \cdot dy}_{\text{area}}$$

We need to express w in terms of y.

y	0	4
w	4	0

so: $w = 4 - y$

The total force on the plate is:

$$\text{Total force} = \int_0^4 (1000)(9.8)(3+y)(4-y)\,dy$$

$$= 9800 \int_0^4 (12 + y - y^2)\,dy$$

$$= 9800 \left[12y + \tfrac{1}{2}y^2 - \tfrac{1}{3}y^3 \right]_0^4$$

$$= 44266.6667 \text{ N.}$$

30. Find the solution of the following initial value problem:

$$y'' + 8y' - 9y = 0 \qquad y(1) = 1 \qquad y'(1) = 0.$$

<u>Characteristic equation:</u> $r^2 + 8r - 9 = 0$, $(r+9)(r-1) = 0$ } Roots are $r = -9$, $r = 1$.

<u>Solution:</u> $y(t) = C_1 e^{-9t} + C_2 e^t$. Determine C_1 and C_2:

<u>$y(1) = 1$:</u> $C_1 e^{-9} + C_2 e = 1$

<u>$y'(1) = 0$:</u> $-9C_1 + C_2 = 0$

So $C_2 = 9C_1$ and

$$C_1 = \frac{1}{e^{-9} + 9e}$$

$$C_2 = \frac{9}{e^{-9} + 9e}.$$

<u>Final answer:</u>

$$y(t) = \frac{1}{e^{-9} + 9e} e^{-9t} + \frac{9}{e^{-9} + 9e} e^t$$