SOLUTIONS

Math 122 **Fall 2008**

Handout 14: Review Problems for Unit Test 3

The topics that will be covered on Unit Test 3 are as follows.

- Calculating formulas for partial sums (e.g. for telescoping series).
- Convergence of infinite series by definition (limit of partial sums).
- Geometric series (finite) and their applications.
- Geometric series (infinite) and their applications.
- n^{th} Term test for divergence.
- Integral test.
- Ratio test.
- Comparison test (compare with p-series or infinite geometric series).
- Alternating series test.
- Absolute versus conditional convergence for alternating series.
- Estimating the sum of an alternating series to a given level of accuracy.
- Summing a finite series with a calculator.
- Finding a formula for the Taylor series of $f(x)$ with center a from the definition.
- Finding a formula for the Taylor series of $f(x)$ with center a by modifying an existing series.
- Radius of convergence of a power series or Taylor series.
- Interval of convergence of a power series or Taylor series.
- Accuracy of Taylor polynomial approximations for functions.

This (roughly) covers Chapter 8 of the textbook.

1. Find the Taylor series of the function $f(x)=\dfrac{x}{\sqrt[3]{1+x^4}}$ centered at $a=0$.

Binomial Theorem:

$$(1+y)^p = 1 + py + \frac{p(p-1)}{2!}y^2 + \frac{p(p-1)(p-2)}{3!}y^3 + \cdots$$

$$f(x) = x\cdot(1+x^4)^{-1/3}$$

$$= x\cdot\left(1 + \frac{-1}{3}x^4 + \frac{(-1/3)(-4/3)}{2!}(x^4)^2 + \frac{(-1/3)(-4/3)(-7/3)}{3!}(x^4)^3 + \cdots\right)$$

$$= x - \frac{1}{3}x^5 + \frac{2}{9}x^9 - \frac{14}{81}x^{13} + \cdots$$

SOLUTIONS

2. Consider the series $\sum_{k=1}^{\infty} \frac{4}{k^2+4k}$. Does this series converge or diverge? If the series converges determine the exact sum of the series.

The series converges. You can show this with the <u>Comparison Test.</u>

Comparison:

$$k^2 + 4k > k^2$$

$$\frac{1}{k^2+4k} < \frac{1}{k^2}$$

$$\frac{4}{k^2+4k} < \frac{4}{k^2}$$

Now, $\sum_{k=1}^{\infty} \frac{4}{k^2}$ is a p-series with $p = 2 > 1$

so it converges. The comparison test gives that $\sum_{k=1}^{\infty} \frac{4}{k^2+4k}$ converges.

Note that $\frac{4}{k^2+4k} = \frac{1}{k} - \frac{1}{k+4}$ so the series will be a telescoping sum. When $N > 3$,

$$S_N = \sum_{k=1}^{N} \frac{4}{k^2+4k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{N+1} - \frac{1}{N+2} - \frac{1}{N+3} - \frac{1}{N+4}$$

The sum of the series is:

$$S = \lim_{N\to\infty} S_N = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$$

SOLUTIONS

3. One of the two sets of functions f_1, f_2, f_3 or g_1, g_2, g_3 is graphed in Figure A; the other set is graphed in Figure B. The Taylor series for the functions about a point corresponding to either A or B are as follows:

$$f_1(x) = a + (x-b) - (x-b)^2 + \ldots \qquad g_1(x) = d - (x-c) - (x-c)^2 + \ldots$$

$$f_2(x) = a - (x-b) + (x-b)^2 + \ldots \qquad g_2(x) = d - (x-c) + (x-c)^2 + \ldots$$

$$f_3(x) = a - 2(x-b) + (x-b)^2 + \ldots \qquad g_3(x) = d + (x-c) + (x-c)^2 + \ldots$$

You may assume that a, b, c and d are all positive constants.

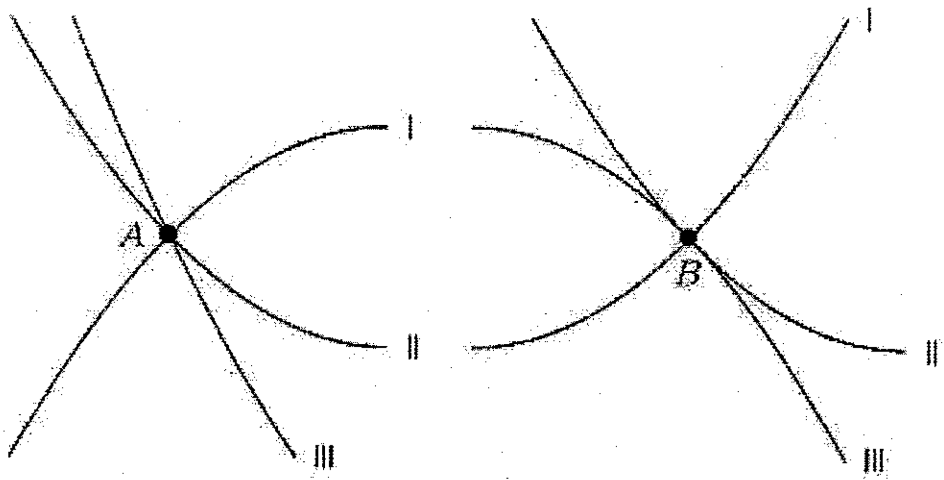

Figure A Figure B

Match each of the graphs shown above with the function $f_1, f_2, f_3, g_1, g_2,$ or g_3 that does the best job of matching the graph. Record your answers in the table below.

Graph	Best matching function	Graph	Best matching function
Figure A, Graph I	$f_1(x)$	Figure B, Graph I	$g_3(x)$
Figure A, Graph II	$f_2(x)$	Figure B, Graph II	$g_2(x)$
Figure A, Graph III	$f_3(x)$	Figure B, Graph III	$g_1(x)$

SOLUTIONS

4. Determine the convergence or divergence of each of the following series. In each case, CIRCLE either CONVERGES or DIVERGES.

In each case, *demonstrate that your answer is correct* step-by-step using *an appropriate convergence test.* Be sure to explicitly state which convergence test you have used. Be careful to show how the convergence test justifies your answer. If you do not justify your answer, you will get zero credit, even if you circle the correct response.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5n+1}{3n^2}$ CONVERGES DIVERGES

JUSTIFICATION: Comparison Test.

Guess that $\displaystyle\sum_{n=1}^{\infty} \frac{5n+1}{3n^2}$ diverges because

for large values of n, $\displaystyle\frac{5n+1}{3n^2} \approx \frac{5n}{3n^2} = \frac{5/3}{n}$

and $\displaystyle\sum_{n=1}^{\infty} \frac{5/3}{n}$ (a p-series with $p=1$) diverges.

Formal comparison: $5n+1 > 5n$

$$\frac{5n+1}{3n^2} > \frac{5n}{3n^2} = \frac{5/3}{n}$$

Now, as mentioned, $\displaystyle\sum_{n=1}^{\infty} \frac{5/3}{n}$ is a p-series with $p=1$ (so it diverges). Since $\displaystyle\frac{5n+1}{3n^2} > \frac{5/3}{n}$, the Comparison Test gives that $\displaystyle\sum_{n=1}^{\infty} \frac{5n+1}{3n^2}$ must diverge also.

(b) $\sum_{n=2}^{\infty} ne^{-n^2}$ CONVERGES DIVERGES

JUSTIFICATION: Integral Test.

Set $f(x) = x \cdot e^{-x^2}$. When $x > 0$,

$$f(x) = \underbrace{x}_{+} \cdot \underbrace{e^{-x^2}}_{+} > 0.$$

Next, $f'(x) = e^{-x^2} - 2x^2 e^{-x^2} = (1-2x^2)e^{-x^2}$.

When $x > 1/\sqrt{2}$, $f'(x) < 0$.

So, the conditions of the Integral test are satisfied.

$$\int_2^{\infty} f(x)\,dx = \lim_{a \to \infty} \int_2^{a} x e^{-x^2}\,dx$$

$$= \lim_{a \to \infty} \left[-\frac{1}{2} e^{-x^2} \right]_2^a$$

$$= \lim_{a \to \infty} -\frac{1}{2} e^{-a^2} + \frac{1}{2} e^{-4}$$

$$= \frac{1}{2} e^{-4}.$$

As the improper integral converges, the Integral test gives that $\sum_{n=2}^{\infty} ne^{-n^2}$ also converges.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{(2n)!}{n!(n+1)!}$ CONVERGES DIVERGES

JUSTIFICATION: Ratio Test.

$$a_n = \frac{(2n)!}{n! \cdot (n+1)!} \qquad a_{n+1} = \frac{(2n+2)!}{(n+1)!\,(n+2)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)!\,(n+2)!} \cdot \frac{n! \cdot (n+1)!}{(2n)!}$$

$$= \frac{(2n+2)(2n+1)}{(n+2)(n+1)}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{4n^2 + 6n + 2}{n^2 + 3n + 1} = 4$$

Since $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, the Ratio test gives that $\displaystyle\sum_{n=2}^{\infty} \frac{(2n)!}{n!\,(n+1)!}$ diverges.

5. **(a)** Find the Taylor Series for the function $f(x)=\dfrac{1}{1+x}$ around the point $a=0$. Express your final answer using sigma (Σ) notation, and clearly indicate your final answer.

$$f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)}$$

$$= \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n \cdot x^n$$

(b) Find the Taylor Series for the function $f(x)=\dfrac{1}{1+x}$ around the point $a=3$. Express your final answer using sigma (Σ) notation, and clearly indicate your final answer.

$$f(x) = \frac{1}{1+x} = \frac{1}{1+3+x-3}$$

$$= \frac{1}{4+(x-3)}$$

$$= \frac{1}{4} \cdot \frac{1}{1-\frac{-1}{4}(x-3)}$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \cdot (x-3)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} \cdot (x-3)^n$$

Continued on the next page.

(c) Find the **interval** of convergence of the Taylor Series for the function $f(x)=\dfrac{1}{1+x}$ around the point $a=0$.

Radius of convergence: Use Ratio test.

$$\frac{a_{n+1}}{a_n} = -x \quad \text{so} \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x| < 1 \quad \text{and}$$

the radius of convergence is $r=1$.

Endpoint $x = 0-1$: Series $= \sum_{n=0}^{\infty} (-1)^n \cdot (-1)^n = \sum_{n=0}^{\infty} 1$

This is a divergent series so $x=-1$ is not in the interval of convergence.

Endpoint $x = 0+1$: Series $= \sum_{n=0}^{\infty} (-1)^n \cdot (1)^n = \sum_{n=0}^{\infty} (-1)^n$

This is a divergent series so $x=1$ is not in the interval of convergence.

Interval of convergence: $(-1, 1)$.

(d) Find the **radius** of convergence of the Taylor Series for the function $f(x)=\dfrac{1}{1+x}$ around the point $a=3$.

Radius of convergence: Use Ratio Test.

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}}{4^{n+2}} \cdot (x-3)^{n+1} \cdot \frac{4^{n+1}}{(-1)^n} \cdot \frac{1}{(x-3)^n}$$

$$= \frac{-1}{4}(x-3)$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}|x-3| < 1 \quad \text{so} \quad |x-3| < 4 .$$

Radius of convergence: $r = 4$