SOLUTIONS

Handout 13: Using Convergence Tests

Determine the convergence or divergence of each of the following series. In each case, demonstrate that your answer is correct in a step-by-step fashion using *an appropriate convergence test.* Be sure to explicitly state which convergence test you have used and show that it can be used with the series you are working on. Be careful to show all of your work and how the convergence test justifies your answer.

(a) Does the following series converge or diverge? $\sum_{n=1}^{\infty} \frac{12(n+1)^2}{4n^2-2n+1}$

CONVERGENCE TEST USED:

The n^{th} term test for divergence.

STEP-BY-STEP JUSTIFICATION:

I think the series diverges because the largest power in the numerator of a_n (n^2) is the same as the largest power of n in the denominator.

To show this I will use the n^{th} term test.

$$a_n = \frac{12n^2 + 24n + 12}{4n^2 - 2n + 1}.$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{12n^2+24n+12}{4n^2-2n+1} = 3.$$

Since $\lim_{n\to\infty} a_n \neq 0$, the n^{th} term test gives that $\sum_{n=1}^{\infty} \frac{12(n+1)^2}{4n^2-2n+1}$ diverges.

FINAL CONCLUSION: CONVERGES DIVERGES

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(b) Does the following series converge or diverge? $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \sqrt{n}}{n+1}$

CONVERGENCE TEST USED:

Alternating series test.

STEP-BY-STEP JUSTIFICATION:

I think the series converges because it is alternating $-+-+-\cdots$ and the largest power of n in the denominator (n^1) is larger than the largest power of n in the numerator ($n^{1/2}$). To show this I will use the alternating series test. Setting $a_n = \frac{\sqrt{n}}{n+1}$, there are two conditions a_n must obey:

(I) $0 \leq a_{n+1} < a_n$ for each value of n

(II) $\lim_{n \to \infty} a_n = 0$.

For condition (I), note: $\frac{d}{dx}\left(\frac{\sqrt{x}}{x+1}\right) = \frac{\frac{1}{2\sqrt{x}}(x+1) - \sqrt{x}}{(x+1)^2}$

$$= \frac{\frac{1}{2\sqrt{x}} - \frac{1}{2}\sqrt{x}}{(x+1)^2} < 0$$

For Condition (II), $\lim_{n \to \infty} \frac{\sqrt{n}}{n+1} = \lim_{n \to \infty} \frac{\sqrt{n}}{n} = 0$.

Since the infinite series satisfies Conditions (I) and (II), the alternating series test gives that $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$ converges.

FINAL CONCLUSION: CONVERGES DIVERGES

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(c) Does the following series converge or diverge? $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

CONVERGENCE TEST USED:

Integral test.

STEP-BY-STEP JUSTIFICATION:

I think this series converges because e^n in the denominator grows much more quickly than n in the numerator.

To show this I will use the integral test with $f(x) = x/e^x = x \cdot e^{-x}$.

To use the integral test we must first verify that for $x > 1$,

(a) $f(x) \geqslant 0$ (b) $f'(x) < 0$

Condition (a): $f(x) = \underbrace{x}_{+} \cdot \underbrace{e^{-x}}_{+} = \text{positive}$

Condition (b): $f'(x) = e^{-x} - x e^{-x} = (1-x) \cdot e^{-x}$

when $x > 1$, $1-x < 0$ making $f'(x) < 0$.

To do the integral test we must calculate:

$$\int_1^\infty x e^{-x} dx = \lim_{a \to \infty} \int_1^a x \cdot e^{-x} dx$$

Integrate by parts $u = x$, $v' = e^{-x}$

$$= \lim_{a \to \infty} \left[-x e^{-x} - e^{-x} \right]_1^a$$

$$= \lim_{a \to \infty} -a e^{-a} - e^{-a} + e^{-1} + e^{-1} = 2e^{-1}$$

Since the improper integral converges, the integral test gives that $\sum_{n=1}^{\infty} n/e^n$ also converges.

FINAL CONCLUSION: CONVERGES DIVERGES

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(d) Does the following series converge or diverge? $\sum_{n=1}^{\infty} \frac{(n!)^2}{1000^n}$

CONVERGENCE TEST USED:

Ratio Test.

STEP-BY-STEP JUSTIFICATION:

I think this series will diverge because even though the 1000^n in the denominator grows rapidly, $(n!)^2$ in the numerator grows even more rapidly.

To show this I will use the ratio test with

$$a_n = \frac{(n!)^2}{1000^n} \quad \text{and} \quad a_{n+1} = \frac{((n+1)!)^2}{1000^{n+1}}.$$

Carrying out the steps in the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2}{1000^{n+1}} \cdot \frac{1000^n}{(n!)^2} = \frac{(n+1)^2}{1000}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{1000} = +\infty.$$

Since this limit is greater than 1, the ratio test gives that $\sum_{n=1}^{\infty} \frac{(n!)^2}{1000^n}$ diverges.

FINAL CONCLUSION: CONVERGES DIVERGES

(d) Does the following series converge or diverge? $\sum_{n=1}^{\infty} \frac{n-3}{n^3+16}$

CONVERGENCE TEST USED:

Comparison test.

STEP-BY-STEP JUSTIFICATION:

I think this series will converge. This is because for very large values of n:

$$\frac{n-3}{n^3+16} \approx \frac{n + \text{peanuts}}{n^3 + \text{peanuts}} \approx \frac{1}{n^2}.$$

Now $\sum_{n=1}^{\infty} 1/n^2$ is a p-series with $p=2$ so it converges. I guess that $\sum_{n=1}^{\infty} \frac{n-3}{n^3+16}$ does something similar.

I will use the comparison test with $a_n = \frac{n-3}{n^3+16}$.

To do the comparison test we must build b_n so that: (I) $a_n \leq b_n$ and (II) $\sum_{n=1}^{\infty} b_n$ converges. Doing this:

$$n^3 + 16 \geq n^3$$

$$\frac{1}{n^3+16} \leq \frac{1}{n^3}$$

Since $n-3 < n$:

$$\frac{n-3}{n^3+16} \leq \frac{n-3}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}.$$

Setting $b_n = 1/n^2$ satisfies (I). To satisfy (II) note that $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n^2$, a p-series with $p=2>1$, so $\sum_{n=1}^{\infty} b_n$ converges. The comparison test then gives that $\sum_{n=1}^{\infty} (n-3)/(n^3+16)$ converges.

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Overall Game Plan for Using Convergence Tests:

When you first see the series, try:

The n^{th} term test for divergence.

If that doesn't work and you notice that the terms of the series alternate between + and –, try:

The alternating series test.

If the first test didn't and the series is not alternating, next try:

The integral test.

- **Use this test also when:**

  a_n corresponds to an $f(x)$ that looks easy to integrate.

- **When using this test, remember to:**

  Check that $f(x) > 0$ and $f'(x) < 0$ over the x-values you will integrate.

If you get a formula for $f(x)$ that is too hard to integrate, try:

The ratio test

- **Also use this when:**

  a_n involves exponentials and/or factorials.

If all else has failed, try:

The comparison test.

- **When doing this, remember the convergence results for p-series:**

  $\sum_{n=1}^{\infty} \frac{c}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

ANSWERS:

(a) Diverge. **(b)** Converge. **(c)** Converge. **(d)** Diverge.
(e) Converge.