Handout 11: Summary of Euler's Method, Slope Fields and Symbolic Solutions of Differential Equations

The purpose of this handout is to review the techniques that you will learn for analyzing the values, graphs and formulas of functions defined by rates of change (differential equations).

- First, we will briefly review **Euler's method** for approximating the values of a function when all that you know is the derivative and one value of the function.

- Second, we will review **slope fields** and **equilibrium solutions** to remember how you can sketch the graph of a function when all that you know is the derivative and one value of the function.

- Finally, we will review the idea of a **symbolic solution** of a differential equation and in particular, how you can decide if a given function is a solution of a differential equation or not and how you can obtain a symbolic solution.

1. A function defined by its rate of change

It is possible to define a function without giving an explicit formula for the function. One way to do this is to specify a function by its rate of change. The two pieces of information that must be given to unambiguously define the function are:

- An equation specifying the rate of change (the derivative) of the function – this is called a **differential equation**, and,

- One value of the function. If this is given for T=0 then this one value of the function is called the **initial value** of the function.

Given such a definition of a function, there are three ways that you can go about figuring out what the values of the function actually are (at least approximately). These are:

- **A numerical approach** such as Euler's method.

- **A graphical approach** such as sketching a slope field.

- **A symbolic approach** in which you try to find a formula for the function.

2. Approximating the values of a function: Euler's method

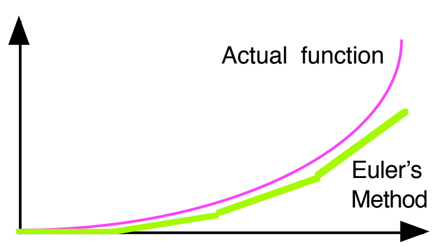
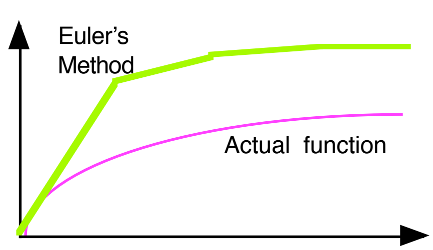
Sometimes we want to approximate the values of a function y(t). However, the function will not always be defined by a convenient formula. Sometimes functions are specified by their rate of change, as is the case for $y(t)$, which is defined by the following equations:

- **A differential equation:** $\frac{dy}{dt} = -y(t) + 1$.
- **An initial value:** $y(0) = 3$.
- **A step-size (often given implicitly in the structure of the table):** $\Delta t = 0.5$.

With this information, the usual way to approximate numerical values of the function is to use Euler's method to construct a table of values.

t	y(t)	$\frac{dy}{dt}$	$\frac{dy}{dt} \cdot \Delta t$	y(t + Δt)
0	3	–2	–1	2
0.5	2	–1	–0.5	1.5
1	1.5	–0.5	–0.25	1.25
Etc.				

Deciding Whether the Function Values from Euler's Method are Over- or Under-Estimates of the True Values of the Function.

Second derivative	Concavity of function	Picture	Numbers from Euler's method are:
Positive (+)	Concave up		Under-estimates of True function values
Negative (–)	Concave down		Over-estimates of True function values

Example:

Consider a function y(t) specified by the following equations:

- **A differential equation:** $\frac{dy}{dt} = -y(t) + 1$**.**
- **An initial value:** y(0) = 3.
- **A step-size (often given implicitly in the structure of the table):** Δt = 0.5.

Is y(1) = 1.5 an over- or an under-estimate of the "true value" of y(1)?

Solution

Starting with the differential equation $\frac{dy}{dt} = -y(t) + 1$ we can calculate the second derivative of the function y(t) by differentiating both sides. Taking the derivative of both sides of the differential equation with respect to the variable 't':

$$\frac{d^2y}{dt^2} = -\frac{dy}{dt} = -(-y(t) + 1) = y(t) - 1.$$

The one (and only) value of the function that we know exactly is y(0) = 3. Substituting this into the second derivative gives:

$$\frac{d^2y}{dt^2} = 3 - 1 = 2.$$

As the second derivative is greater than zero, the function y(t) is concave up and the function values produced by Euler's method will be **under-estimates** of the "true values" of the function.

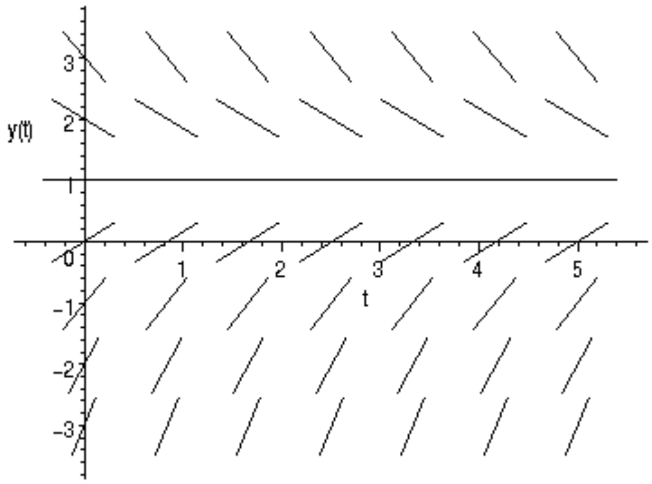
3. Sketching the graph of a function: Slope fields and equilibrium solutions

You can think of the slope field as a collection of guidelines that help you shape the graph of a function. To draw a slope field all that you need is a differential equation.

Example:

Draw the slope field defined by the differential equation: $\frac{dy}{dt} = -y(t) + 1.$

Solution

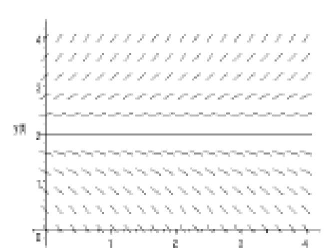


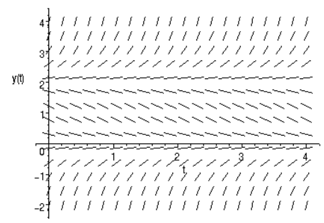
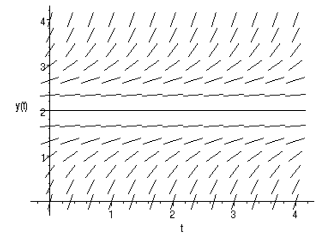
Equilibrium Solutions of a Differential Equation

The **equilibrium solutions** of a differential equation are those curves (often horizontal lines) along which the derivative is equal to zero.

There are three main types of equilibrium solution. These are:

- **Stable equilibriums:** Line segments are attracted towards the equilibrium from above and below.
- **Unstable equilibriums:** Line segments are repelled away from the equilibrium above and below.
- **Semi-stable equilibriums:** Line segments are attracted from one side and repelled on the other side.

Equation for Derivative	Equilibrium solution(s)	Appearance of Slope field	Nature of Equilibrium solution(s)
$y'(t) = y(t) - 2$	• y(t) = 2		y(t) = 2 is an unstable equilibrium

$y'(t) = y(t) \cdot [y(t) - 2]$	• y(t) = 0 • y(t) = 2		y(t) = 0 is a stable equilibrium y(t) = 2 is an unstable equilibrium
$y'(t) = [y(t) - 2]^2$	• y(t) = 2		y(t) = 2 is a semi-stable equilibrium

4. The exact formula for the function: Symbolic solutions

A **symbolic solution** for a differential equation is an explicit formula for the function that is defined by the differential equation and initial value.

To check that a given function is actually a symbolic solution of a given differential equation and initial value you can:

i. Substitute the formula for the function into the **left hand side** of the differential equation.

ii. Substitute the formula for the function into the **right hand side** of the differential equation.

iii. If the results that you obtain from substituting the formula into the left and the right hand sides are the same, then the function is a solution of the differential equation.

iv. If you plug t=0 into the formula for the function and you obtain the initial value as a result, then the function also satisfies the initial value.

Example:

Consider the differential equation and initial condition: $\frac{dy}{dt} = -y(t) + 1$

and initial condition: y(0) = 3.

Verify that the function $y(t) = 1 + 2\cdot e^{-t}$ is a symbolic solution of the differential equation and initial condition.

Solution

a. Verifying that the Function Satisfies the Differential Equation:

$$
\begin{aligned}
\textit{Left hand side} &= \frac{dy}{dt} \\
&= -2\cdot e^{-t}. \\
\textit{Right hand side} &= -y(t) + 1 \\
&= -(1 + 2\cdot e^{-t}) + 1 \\
&= -2\cdot e^{-t}.
\end{aligned}
$$

As the left and right hand sides are equal, the given function satisfies the differential equation.

b. Verifying that the Function Satisfies the Initial Condition:

$$y(0) = 1 + 2\cdot e^{0} = 1 + 2 = 3.$$

5. Generating Symbolic Solutions: Separation of Variables

A **symbolic solution** for a differential equation is an explicit formula for the function that is defined by the differential equation and initial value.

The point of doing the technique of Separation of variables is always to find an explicit formula for a function, such as y(t).

i. **Rewrite the differential equation to make Separation of Variables as easy as possible.** Replace $y'(t)$ by $\frac{dy}{dt}$ and $y(t)$ by just y.

ii. **Rearrange the differential equation to get all of the *y*'s on one side and all of the *t*'s on the other side.**

iii. **Integrate both sides of the equation with respect to the appropriate variable.**

iv. **Rearrange to make *y* the subject of the equation.**

v. **Use the initial condition to find the numerical value of the constant.**

Example: Separation of Variables

Find a formula for the function $y(t)$ defined by the following differential equation and initial value:

Differential equation: $y'(t) = 3 \cdot [y(t) - 1]$

Initial value: $y(0) = 5$.

Solution:

i. **Rewrite the differential equation to make Separation of Variables as easy as possible.** Replace $y'(t)$ by $\frac{dy}{dt}$ and $y(t)$ by just y.

$$\frac{dy}{dt} = 3 \cdot [y - 1]$$

ii. **Rearrange the differential equation to get all of the *y*'s on one side and all of the *t*'s on the other side.**

$$\frac{dy}{y-1} = 3 \cdot dt$$

iii. **Integrate both sides of the equation with respect to the appropriate variable.**

$$\int \frac{dy}{y-1} = \int 3 \cdot dt$$

$$\ln(y-1) = 3 \cdot t + C$$

iv. **Rearrange to make *y* the subject of the equation.**

$$y - 1 = e^{3t+C} = e^{C} \cdot e^{3t} = A \cdot e^{3t}$$

$$y = 1 + A \cdot e^{3t}$$

v. **Use the initial condition to find the numerical value of the constant.**

$$5 = 1 + A \cdot e^{0}$$

$$A = 4$$

$$y = 1 + 4 \cdot e^{3t}$$

Example 13-2b: Separation of Variables

Find a formula for the function *P*(*t*) defined by the following differential equation and initial value:

Differential equation: $P'(t) = 2 - 0.1 \cdot P(t)$
Initial value: $P(0) = 1$.

Solution:

i. **Rewrite the differential equation to make Separation of Variables as easy as possible.** Replace $y'(t)$ by $\frac{dy}{dt}$ and *y*(*t*) by just *y*.

$$\frac{dP}{dt} = -0.1 \cdot [P - 20]$$

ii. **Rearrange the differential equation to get all of the *y*'s on one side and all of the *t*'s on the other side.**

$$\frac{dP}{P - 20} = -0.1 \cdot dt$$

iii. **Integrate both sides of the equation with respect to the appropriate variable.**

$$\int \frac{dP}{P - 20} = \int -0.1 \cdot dt$$

$$\ln(P - 20) = -0.1 \cdot t + C$$

iv. **Rearrange to make *y* the subject of the equation.**

$$P - 20 = e^{-0.1t + C} = e^{C} \cdot e^{-0.1t} = A \cdot e^{-0.1t}$$

$$y = 20 + A \cdot e^{-0.1t}$$

v. **Use the initial condition to find the numerical value of the constant.**

$$1 = 20 + A \cdot e^{0}$$

$$A = -19$$

$$P = 20 - 19 \cdot e^{-0.1t}$$