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REMARKS ON A PAPER BY CHAO-PING CHEN AND FENG QI

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ABSTRACT. In a recent paper, Chao-Ping Chen and Feng Qi (2005) established sharp upper and lower bounds for the sequence $P_n := \frac{1.3...(2n-1)}{2.4...2n}$. We show that their result follows easily from a theorem of G. N Watson published in 1959. We also show that the main result of Chen and Qi's paper is a special case of a more general inequality which admits a very short proof.

1. INTRODUCTION AND RESULTS

Let

$$P_n := \frac{1.3\dots(2n-1)}{2.4\dots 2n}.$$

This sequence appears in Wallis's well-known product formula of approximation of π and in various other topics of analysis and number theory. In the recent paper [2], Chao-Ping Chen and Feng Qi proved the following inequality:

$$\frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}} \le P_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}, \quad n = 1, 2, \dots. \tag{1.1}$$

The constants $\frac{4}{\pi} - 1$ and $\frac{1}{4}$ are the best possible. Inequality (1.1) improves some earlier results dealing with estimates of the sequence P_n.

Here we observe that (1.1) follows easily from a known result due to G. N. Watson [15]. Indeed, it is shown in [15] that the function

$$\theta(x) = \left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right)^2 - x$$

is strictly decreasing on $(-1/2, \infty)$. Applying this result, together with the observation that $\theta(1) = 4/\pi - 1$, $\lim_{x\to\infty} \theta(x) = 1/4$ and $P_n = \dfrac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)}$, we obtain the double inequality (1.1).

The purpose of this note is to show that (1.1) is a special case of a more general inequality, which in fact, admits a very short proof. Let $0 < \alpha < 1$ and

$$d_n(\alpha) := \frac{(1-\alpha)_n}{n!}\,.$$

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As usual, $(a)_k$ denotes the Pochhammer symbol, defined by

$$(a)_0 = 1, \text{ and } (a)_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(k+a)}{\Gamma(a)}, \text{ for } k = 1, 2, \dots.$$

Clearly, $d_n(1/2) = P_n$. It is also well known and easy to see that

$$\frac{1}{\Gamma(1-\alpha)\,(n+1)^\alpha} < d_n(\alpha) < \frac{1}{\Gamma(1-\alpha)\,n^\alpha}. \tag{1.2}$$

Here we show that this inequality can be improved as follows.

Theorem 1.1. *For all natural numbers n, we have*

$$\frac{1}{\Gamma(1-\alpha)\,(n+c_2)^\alpha} \le d_n(\alpha) < \frac{1}{\Gamma(1-\alpha)\,(n+c_1)^\alpha}\,, \tag{1.3}$$

where the constants

$$c_1 = c_1(\alpha) = \frac{1-\alpha}{2} \quad \text{and} \quad c_2 = c_2(\alpha) = \frac{1}{\big[\Gamma(2-\alpha)\big]^{1/\alpha}} - 1$$

are the best possible.

It is clear that for $\alpha = 1/2$, inequality (1.3) coincides with (1.1).

Inequalities such as (1.2) and (1.3) are of particular importance in certain problems on positive trigonometric sums and positive sums of Gegenbauer polynomials, having $d_k(\alpha)$ as a sequence of coefficients. See the recent papers [9] and [10]. We also refer to the paper [11] for some interesting estimates of ratios of gamma functions. Some other proofs of (1.1) and related results can be found in the recent articles [3], [4], [5], [6] and [7]. Compare also the papers [12], [13] and [14] for different proofs of (1.1) and various inequalities for the sequence P_n.

2. Proof of (1.3)

For the proof of (1.3) we define

$$Q_n(\alpha) := \left\{ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} \right\}^{1/\alpha} - n\,.$$

Using the asymptotic formula

$$x^{b-a}\,\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\Big(\frac{1}{x^2}\Big), \quad x \to \infty$$

(see [1], p. 615, for the complete form of this formula), we easily verify that

$$\lim_{n\to\infty} Q_n(\alpha) = \frac{1-\alpha}{2} = c_1.$$

Obviously $c_2 = Q(1, \alpha)$.

The required inequality (1.3) follows from the fact that the sequence $Q_n(\alpha)$ is strictly decreasing. This, in turn, is an immediate consequence of a result of N. Elezović, C. Giordano and J. Pečarić, [8] who showed that the function

$$x \mapsto \Big(\frac{\Gamma(x+t)}{\Gamma(x+s)}\Big)^{1/(t-s)} - x$$

is convex and decreasing on $(-r, \infty)$ if $|t-s| < 1$, where s, t are given positive numbers and $r = \min(s, t)$.

This completes the proof of (1.3). $\square$

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