MATH 122 – THIRD UNIT TEST

Friday, November 21, 2008.

NAME: SOLUTIONS

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A	B	D	F	H	J
	C	E	G	I	K

Instructions:

1. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
2. Please read the instructions for each individual question carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated. However, if you are asked to find an ***exact*** value of a quantity that involves an integral then you should not use calculator integration for this.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. If you evaluate an improper integral, be sure to use appropriate algebraic and limit notation.
7. Please **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

Problem	Total	Score
1	15	
2	20	
3	30	
4	20	
5	15	
Total	100	

SOLUTIONS

1. 15 Points. SHOW ALL WORK. NO WORK = NO CREDIT.

Consider the infinite series:

$$\sum_{k=1}^{\infty} \frac{3}{k^2+3k}.$$

(a) **(4 points)** Use the Technique of Partial Fractions to write $\frac{3}{k^2+3k}$ as the difference of two simpler fractions. Your final answer should contain no unspecified quantities beyond k.

$$\frac{3}{k^2+3k} = \frac{3}{k(k+3)} = \frac{A}{k} - \frac{B}{k+3}$$

Equations for A and B: $\left.\begin{array}{rl} A - B &= 0 \\ 3A &= 3 \end{array}\right\}$ $A = B = 1.$

Final answer: $\frac{3}{k^2+3k} = \frac{1}{k} - \frac{1}{k+3}$

(b) **(3 points)** Calculate the partial sum indicated below. Include at least **8 decimal places** in your answer.

$$\sum_{k=1}^{5} \frac{3}{k^2+3k} = \frac{1}{1} - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{7} + \frac{1}{5} - \frac{1}{8}$$

$$= 1.398809524$$

(c) **(4 points)** Find a simple formula that gives the N^{th} partial sum of the infinite series that is valid for $N > 2$. The only variable in your formula should be N.

$$S_N = \sum_{k=1}^{N} \frac{3}{k^2+3k} = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{N+1} - \frac{1}{N+2} - \frac{1}{N+3}$$

(d) **(4 points)** Find the **exact** sum of the infinite series $\sum_{k=1}^{\infty} \frac{3}{k^2+3k}$. If you express your final answer as a decimal, include at least **8 decimal places**.

$$\sum_{k=1}^{\infty} \frac{3}{k^2+3k} = \lim_{N \to \infty} S_N = 1 + \frac{1}{2} + \frac{1}{3} \doteq 1.83333333$$

2. 20 Points. SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.

The function $f(x)$ has a Taylor series that converges to $f(x)$ for all x in the interval of convergence. The n^{th} derivative of $f(x)$ evaluated at $a = 10$ is:

$$f^{(n)}(a) = f^{(n)}(10) = \frac{(-1)^n \cdot (n-1)!}{2^n} \qquad \text{for all } n \geq 1.$$

The value of the function at the point $a = 10$ is: $f(a) = f(10) = -1$.

(a) **(4 points)** Write down the Taylor series of the function $f(x)$ about $a = 10$. To express your answer, use Σ notation.

$$-1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 2^n} (x-10)^n$$

(b) **(8 points)** Find the **radius** of convergence for the Taylor series of $f(x)$ about $a = 10$. Show your work.

Use the Ratio Test with $a_n = \dfrac{(-1)^n}{n \cdot 2^n} \cdot (x-10)^n$.

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}}{(n+1) \cdot 2^{n+1}} (x-10)^{n+1} \cdot \frac{n \cdot 2^n}{(-1)^n} \cdot \frac{1}{(x-10)^n}$$

$$= \frac{-n}{(n+1) \cdot 2} (x-10)$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{n}{2(n+1)} |x-10| = \frac{1}{2} |x-10| < 1$$

$$\therefore \quad |x-10| < 2.$$

The radius of convergence is $r = 2$.

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The function $f(x)$ has a Taylor series that converges to $f(x)$ for all x in the interval of convergence. The n^{th} derivative of $f(x)$ evaluated at $a = 10$ is:

$$f^{(n)}(a) = f^{(n)}(10) = \frac{(-1)^n \cdot (n-1)!}{2^n} \qquad \text{for all } n \geq 1.$$

The value of the function at the point $a = 10$ is: $f(a) = f(10) = -1$.

(c) **(8 points)** Find the **interval** of convergence for the Taylor series of $f(x)$ about $a = 10$. Show your work and clearly indicate your final answer. No work = no credit.

The Taylor series is centered at $a = 10$. The radius of convergence is $r = 2$. The interval of convergence includes:

$$8 < x < 12.$$

We must investigate convergence at the endpoints.

$\underline{x = 8:}$ Plugging $x = 8$ into the Taylor series gives:

$$-1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 2^n} \cdot (8-10)^n = -1 + \sum_{n=1}^{\infty} \frac{1}{n}$$

Now, $\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1$ so it diverges.

So the Taylor series does not converge at $x = 8$.

$\underline{x = 12:}$ Plugging $x = 12$ into the Taylor series gives:

$$-1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 2^n} \cdot (12-10)^n = -1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

As $\frac{1}{n+1} < \frac{1}{n}$ and $\lim_{n \to \infty} \frac{1}{n} = 0$ the Alternating Series Test gives that $-1 + \sum_{n=1}^{\infty} (-1)^n/n$ converges so the Taylor series converges at $x = 12$.

$\underline{\text{Interval of Convergence:}}$ $(8, 12]$.

SOLUTIONS

3. 30 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.

Determine the convergence or divergence of each of the following series. If you do not justify your answer, you will get zero credit, even if you circle the correct final answer.

In each case, demonstrate that your answer is correct in a step-by-step fashion using *an appropriate convergence test*. Be sure to explicitly state which convergence test you have used and show that it can be used with the series you are working on. Be careful to show all of your work. As the final part of your answer in each part, CIRCLE either CONVERGES or DIVERGES.

(a) (10 points) $$\sum_{n=1}^{\infty} \frac{e^n}{n!}$$

JUSTIFICATION: We will use the Ratio Test.

$$a_n = \frac{e^n}{n!} \qquad a_{n+1} = \frac{e^{n+1}}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} = \frac{e}{n+1}.$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{e}{n+1} = 0.$$

As the limit of the ratio is less than 1, the Ratio test gives that the series $\sum_{n=1}^{\infty} \frac{e^n}{n!}$ converges.

FINAL ANSWER (CIRCLE ONE): CONVERGES DIVERGES

Demonstrate that your answer is correct in a step-by-step fashion using *an appropriate convergence test.* Be sure to explicitly state which convergence test you have used and show that it can be used with the series you are working on. Be careful to show all of your work. As the final part of your answer in each part, CIRCLE either CONVERGES or DIVERGES.

(b) **(10 points)** $\sum_{n=1}^{\infty} n \cdot e^{-n^2}$

JUSTIFICATION: We will use the Integral Test.

Set $f(x) = x \cdot e^{-x^2}$.

Conditions on $f(x)$: (I) When $x > 1$, $f(x) = \underbrace{x}_{+} \cdot \underbrace{e^{-x^2}}_{+} > 0$.

(II) $f'(x) = \underbrace{(1 - 2x^2)}_{-} \cdot \underbrace{e^{-x^2}}_{+} < 0$ when $x > 1$.

Improper Integral:

$$\int_1^\infty f(x)\,dx = \int_1^\infty x \cdot e^{-x^2} \cdot dx$$

$$= \lim_{a \to \infty} \int_1^a x \cdot e^{-x^2} \cdot dx$$

$$= \lim_{a \to \infty} \left[\frac{-1}{2} e^{-x^2} \right]_1^a$$

$$= \lim_{a \to \infty} -\frac{1}{2} e^{-a^2} + \frac{1}{2} e^{-1} = \frac{1}{2} e^{-1}$$

As the improper integral converges, the Integral test gives that the series $\sum_{n=1}^{\infty} n \cdot e^{-n^2}$ also converges.

FINAL ANSWER (CIRCLE ONE): CONVERGES DIVERGES

Demonstrate that your answer is correct in a step-by-step fashion using *an appropriate convergence test.* Be sure to explicitly state which convergence test you have used and show that it can be used with the series you are working on. Be careful to show all of your work. As the final part of your answer in each part, CIRCLE either CONVERGES or DIVERGES.

(c) **(10 points)** $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

JUSTIFICATION: We will use the Comparison Test.

Guess: I guess that $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges. This is because for large values of N,

$$\frac{\sqrt{n}}{n^2+1} \approx \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

and $\sum_{n=1}^{\infty} 1/n^{3/2}$ is a p-series with $p = 3/2 > 1$ (hence convergent).

Formal Comparison:

$$n^2+1 > n^2$$

$$\frac{1}{n^2+1} < \frac{1}{n^2}$$

$$\frac{\sqrt{n}}{n^2+1} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

As noted above $\sum_{n=1}^{\infty} 1/n^{3/2}$ is a p-series with $p = 3/2 > 1$ (hence convergent). As $\frac{\sqrt{n}}{n^2+1} < \frac{1}{n^{3/2}}$ the Comparison Test gives that $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges.

FINAL ANSWER (CIRCLE ONE): CONVERGES DIVERGES

4. **20 Points. SHOW YOUR WORK.**

Consider the alternating series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}.$$

(a) **(10 points)** Does the series converge absolutely, converge conditionally or diverge? Clearly state your answer and use a convergence test to demonstrate that your answer is correct.

The series converges absolutely (and hence also conditionally). This can be demonstrated using the <u>Ratio Test</u>.

$$a_n = \frac{(-1)^n}{(2n)!} \qquad a_{n+1} = \frac{(-1)^{n+1}}{(2n+2)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n} = \frac{-1}{(2n+2)(2n+1)}$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{1}{(2n+2)(2n+1)} = 0.$$

As the limit of the ratio is less than one the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$ converges absolutely by the Ratio test.

Continued on the next page.

The alternating series from the previous page is:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}.$$

(b) **(6 points)** Let S represent the sum of the series. Suppose that S, the sum of the series, is approximated by the N^{th} partial sum:

$$S_N = \sum_{n=1}^{N} \frac{(-1)^n}{(2n)!}.$$

What is the smallest value of N that could be used to approximate S by S_N and an error of less than 0.0001?

Set $a_n = \dfrac{1}{(2n)!}$. As the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$ is convergent, the Alternating Series Estimation Theorem gives:

$$|S - S_N| < a_{N+1}$$

where $S = \sum_{n=1}^{\infty} (-1)^n / (2n)!$. To solve the problem we must solve the following equation for N:

$$a_{N+1} = \frac{1}{(2N+2)!} < 0.0001.$$

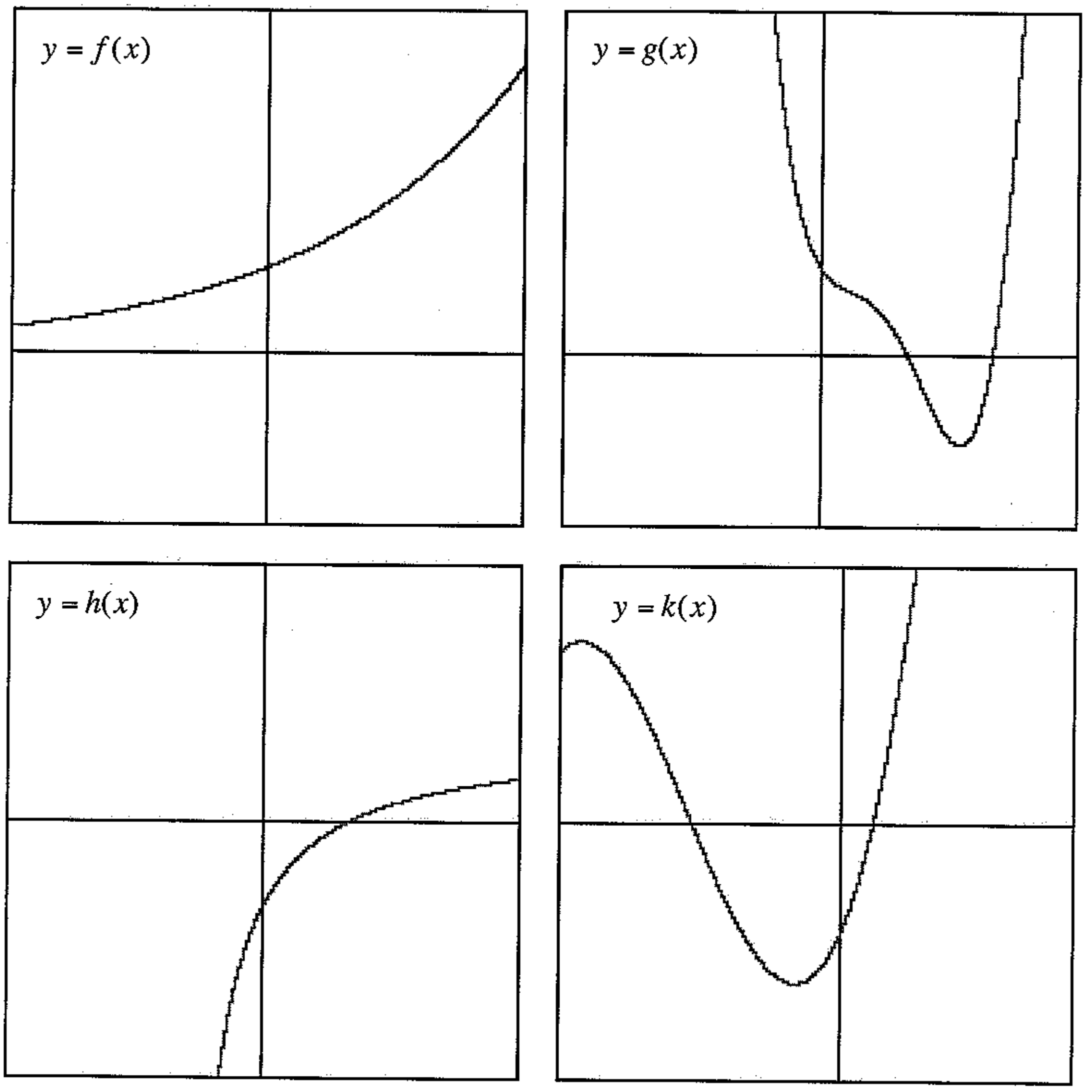
Using tables on a calculator gives: $\boxed{N \geq 3.}$
$N = 3$ is the smallest value.

(c) **(4 points)** Approximate the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$ with an error of less than 0.0001. Include at least **8 decimal places** in your answer.

$$\sum_{n=1}^{3} \frac{(-1)^n}{(2n)!} = -0.45972222$$

5. 15 Points. 3 POINTS FOR EACH CORRECT RESPONSE.

The diagram shows the graphs of four functions, $f(x)$, $g(x)$, $h(x)$ and $k(x)$.



The formulas given below are Taylor polynomials of degree 2 centered at zero (a = 0). Match the Taylor polynomials with the functions. If you believe that a particular Taylor polynomial could not possibly correspond to any of the functions $f(x)$, $g(x)$, $h(x)$ or $k(x)$ write "**NONE**" in the space provided beside it.

(a) $P_2(x) = 1 + x + x^2$ Corresponding function: $f(x)$

(b) $P_2(x) = -1 + x + x^2$ Corresponding function: $k(x)$

(c) $P_2(x) = 1 - x + x^2$ Corresponding function: $g(x)$

(d) $P_2(x) = -1 + x - x^2$ Corresponding function: $h(x)$

(e) $P_2(x) = -1 - x + x^2$ Corresponding function: NONE