MATH 122 – FIRST UNIT TEST

Friday, September 19, 2008.

NAME: SOLUTIONS ID NUMBER: ____________

RECITATION SECTION: ____________ RECITATION TA: ____________

Instructions:

1. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
2. Please read the instructions for each individual question carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated. However, if you are asked to find an ***exact*** value of a quantity that involves an integral then you should not use calculator integration for this.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. If you evaluate an improper integral, be sure to use appropriate algebraic and limit notation.
7. Please **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

Problem	Total	Score
1	16	
2	20	
3	30	
4	20	
5	15	
Total	100	

SOLUTIONS

1. 16 Points. SHOW ALL WORK. NO WORK = NO CREDIT.

Evaluate each of the indefinite integrals to find the most general antiderivative. In each case, show your work and clearly indicate your final answer. No work, no credit even if your final answer is correct.

You may use the following indefinite integral formula without having to justify it:

$$\int \sec(x)dx = \ln(|\sec(x) + \tan(x)|) + C.$$

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

(a) **(8 points)** $\int \tan^3(x)\sec^3(x)dx$

$$\int \tan^3(x)\sec^3(x)\,dx = \int \tan(x)\sec(x)\tan^2(x)\sec^2(x)\,dx$$

Use $\tan^2(x) = \sec^2(x) - 1$ to rewrite this as:

$$\int \tan^3(x)\sec^3(x)\,dx = \int \tan(x)\sec(x)\left(\sec^2(x)-1\right)\sec^2(x)\,dx$$

$$= \int \tan(x)\cdot\sec(x)\cdot\sec^4(x)\,dx$$

$$- \int \tan(x)\cdot\sec(x)\cdot\sec^2(x)\,dx$$

Now make the substitution $u = \sec(x)$ and $du = \sec(x)\cdot\tan(x)\,dx$ to obtain:

$$\int \tan^3(x)\sec^3(x)\,dx = \int u^4\,du - \int u^2\,du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$= \frac{1}{5}\sec^5(x) - \frac{1}{3}\sec^3(x) + C.$$

Continued on the next page.

SOLUTIONS

Evaluate each of the indefinite integrals to find the most general antiderivative. In each case, show your work and clearly indicate your final answer. No work, no credit even if your final answer is correct.

You may use the following indefinite integral formula without having to justify it:

$$\int \sec(x)dx = \ln(|\sec(x)+\tan(x)|)+C.$$

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

(b) **(8 points)** $\int \sec^3(w)dw$

Note that $\sec^3(w) = \sec(w)\cdot\sec^2(w)$. We will integrate by parts using:

$u = \sec(w)$ $\quad v' = \sec^2(w)$

$u' = \sec(w)\cdot\tan(w)$ $\quad v = \tan(w)$

giving:

$$\int \sec^3(w)\,dw = \sec(w)\cdot\tan(w) - \int \sec(w)\cdot\tan^2(w)\,dw$$

Next use the identity $\tan^2(w) = \sec^2(w) - 1$ to obtain:

$$\int \sec^3(w)\,dw = \sec(w)\cdot\tan(w) - \int \sec(w)\cdot(\sec^2(w)-1)\cdot dw$$

$$= \sec(w)\cdot\tan(w) - \int \sec^3(w)\cdot dw + \int \sec(w)\,dw$$

Adding $\int \sec^3(w)\,dw$ to both sides and then dividing by 2:

$$\int \sec^3(w)\,dw = \frac{1}{2}\sec(w)\cdot\tan(w) + \frac{1}{2}\int \sec(w)\,dw$$

$$= \frac{1}{2}\sec(w)\cdot\tan(w) + \frac{1}{2}\ln(|\sec(w)+\tan(w)|) + C$$

using the formula given above.

SOLUTIONS

2. 20 Points. SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.

In this problem, all that you may assume that $f(x)$ is an **odd** function, that $g(x)$ is an **even** function, that the domains of both functions include $-4 \le x \le 4$ and that they have the values shown below.

$$f(-2)=3 \qquad f(0)=0 \qquad g(2)=-2 \qquad g'(2)=4.$$

(a) **(5 points)** Find the exact value of the definite integral: $\int_{-3}^{3} f(x)+f(x)\cdot g(x)dx$.

$$\int_{-3}^{3} f(x) + f(x)\cdot g(x)\,dx = \int_{-3}^{3} \underbrace{f(x)}_{\text{odd}}\,dx + \int_{-3}^{3} \underbrace{f(x)\cdot g(x)}_{\text{odd}\cdot\text{even} = \text{odd}}\,dx$$

$$= 0.$$

(b) **(5 points)** Find the exact value of the definite integral: $\int_{-2}^{2} x\cdot f'(x)dx$.

$$\int_{-2}^{2} x\cdot f'(x)\,dx = \left[x\cdot f(x)\right]_{-2}^{2} - \int_{-2}^{2} \underbrace{f(x)}_{\text{odd}}\,dx$$

$$= 2\cdot f(2) - (-2)\cdot f(-2) = 0.$$

(c) **(5 points)** Find the exact value of the definite integral: $\int_{0}^{2} f'(x)\cdot f(x)^3 dx$.

$$\int_{0}^{2} f'(x)\cdot f(x)^3\cdot dx = \left[\tfrac{1}{4} f(x)^4\right]_{0}^{2} = \tfrac{1}{4} f(2)^4$$

$$= \tfrac{1}{4}(-3)^4 = \frac{81}{4}$$

(d) **(5 points)** Find the exact value of the definite integral: $\int_{0}^{\sqrt{2}} x\cdot f'(x^2)dx$.

$$\int_{0}^{\sqrt{2}} x\cdot f'(x^2)\,dx = \tfrac{1}{2}\int_{0}^{2} f'(u)\,du = \tfrac{1}{2}\left[f(u)\right]_{0}^{2} = \frac{-3}{2}$$

Substitute $u = x^2$.

3. 30 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable. No work = no credit even if your answer is correct. Your answers may contain unspecified constants (such as M, N, B and R).

You may use the following indefinite integral formula without having to justify it:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C.$$

(a) **(10 points)** Find a formula for the most general antiderivative of: $f(x)=\dfrac{1}{(x-2M)(x+N)}$ where M and N are constants and $N+2M\neq 0$.

Use partial fractions: $\dfrac{1}{(x-2M)(x+N)} = \dfrac{A}{x-2M} + \dfrac{B}{x+N}$.

To calculate A and B, add fractions and equate powers of x in numerators.

$$A\cdot(x+N) + B\cdot(x-2M) = 1$$

Coefficient of x: $A + B = 0$

Constants: $A\cdot N - 2B\cdot M = 1$

Solving for A and B in terms of M and N gives:

$$A = \frac{1}{N+2M} \qquad B = \frac{-1}{N+2M}$$

so that:

$$\int f(x)\,dx = \frac{1}{N+2M}\ln(|x-2M|) - \frac{1}{N+2M}\ln(|x+N|) + C.$$

Continued on the next page.

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable. No work = no credit even if your answer is correct. Your answers may contain unspecified constants (such as M, N, B and R).

You may use the following indefinite integral formula without having to justify it:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C.$$

(b) **(10 points)** Find a formula for the most general antiderivative of: $g(x)=\dfrac{3}{x^2-2Bx+2B^2}$ where B is a positive constant.

Use completing the square.

$$x^2 - 2Bx + 2B^2 = (x-B)^2 + B^2$$

Using the integration formula given above and the substitution $u = x - B$:

$$\int g(x)\,dx = \int \frac{3}{(x-B)^2+B^2}\,dx$$

$$= 3\cdot\int \frac{1}{u^2+B^2}\,du$$

$$= \frac{3}{B}\arctan\left(\frac{u}{B}\right)+C$$

$$= \frac{3}{B}\arctan\left(\frac{x-B}{B}\right)+C$$

Continued on the next page.

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable. No work = no credit even if your answer is correct. Your answers may contain unspecified constants (such as M, N, B and R).

You may use the following indefinite integral formula without having to justify it:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$$

(c) **(10 points)** Find a formula for the most general antiderivative of: $j(x) = \dfrac{x^3 + Rx^2 - R^2x - R^3}{x - R}$ where R is a positive constant.

Use polynomial long division.

$$\begin{array}{r|llll}
 & x^2 & +\,2Rx & +\,R^2 & \\
\hline
x - R & x^3 & +\,Rx^2 & -\,R^2x & -\,R^3 \\
 & x^3 & -\,Rx^2 & & \\
\cline{2-3}
 & & 2Rx^2 & -\,R^2x & \\
 & & 2Rx^2 & -\,2R^2x & \\
\cline{3-4}
 & & & R^2x & -\,R^3 \\
 & & & R^2x & -\,R^3 \\
\cline{4-5}
 & & & & 0
\end{array}$$

So:

$$\int j(x)\,dx = \int x^2 + 2Rx + R^2\,dx$$

$$= \frac{1}{3}x^3 + Rx^2 + R^2x + C.$$

SOLUTIONS

4. **20 Points. INCLUDE ALL DECIMAL PLACES. SHOW YOUR WORK IN PART (D).**

(a) **(4 points)** Find the approximate value of $\int_1^4 e^{\sqrt{x}}dx$ obtained when the integral is approximated using the Midpoint Rule and 50 rectangles.

Midpoint $= 14.77803897$

(b) **(4 points)** Find the approximate value of $\int_1^4 e^{\sqrt{x}}dx$ obtained when the integral is approximated using the Trapezoid Rule and 50 trapezoids.

Trapezoid $= 14.77825864$

(c) **(4 points)** Find the approximate value of $\int_1^4 e^{\sqrt{x}}dx$ obtained when the integral is approximated using Simpson's Rule and 100 rectangles.

Simpson $= 14.7781122$

(d) **(8 points)** How many rectangles should you use if you wanted to approximate the value of $\int_1^4 e^{\sqrt{x}}dx$ using the Midpoint Rule and with an error of less than 0.001? Show your work.

For midpoint rule: $|\text{Error}| \le \dfrac{K\cdot(b-a)^3}{24\cdot N^2}$

$f(x) = e^{\sqrt{x}}$ $\qquad f'(x) = \dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}$

$f''(x) = \dfrac{1}{4x}e^{\sqrt{x}} - \dfrac{1}{4x^{3/2}}e^{\sqrt{x}}.$

Graphing $|f''(x)|$ on the interval $[1,4]$ gives $K = 0.231$

To find N: $\dfrac{(0.231)(4-1)^3}{24\cdot N^2} < 0.001$

Solving for N: $N > \sqrt{\dfrac{(0.231)(4-1)^3}{(24)(0.001)}} = 16.12$

Use at least 17 rectangles.

5. 15 Points. SHOW ALL WORK. NO WORK = NO CREDIT.

Use the technique of Trigonometric substitution to evaluate the definite integral shown below. Show your work and clearly indicate your final answer. No work = no credit even if your answer is correct.

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

$$\int_1^2 \frac{1}{x^2\sqrt{x^2+9}}\,dx$$

Use trigonometric substitution with $x = 3 \cdot \tan(\theta)$.
Then $dx = 3 \cdot \sec^2(\theta)\,d\theta$ and $x^2 + 9 = 9 \cdot \sec^2(\theta)$.

Carrying out the substitution:

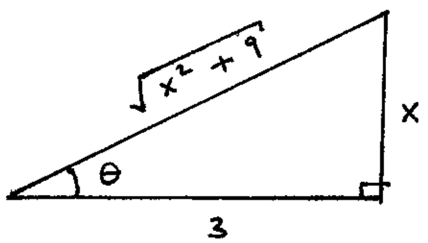
$$\int \frac{1}{x^2\sqrt{x^2+9}}\,dx = \int \frac{1}{9 \cdot \tan^2(\theta) \cdot 3 \cdot \sec(\theta)} \cdot 3\sec^2(\theta) \cdot d\theta$$

$$= \frac{1}{9}\int \frac{\sec(\theta)}{\tan^2(\theta)}\,d\theta$$

Now, $\frac{\sec(\theta)}{\tan^2(\theta)} = \frac{\cos(\theta)}{\sin^2(\theta)}$ which can be integrated via the u-substitution $u = \sin(\theta)$. This gives:

$$\int \frac{1}{x^2\sqrt{x^2+9}}\,dx = \frac{1}{9}\int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta = \frac{-1}{9}\csc(\theta) + C$$

Converting the antiderivative back to "x":



so $\csc(\theta) = \frac{\sqrt{x^2+9}}{x}$ and

$$\int \frac{1}{x^2\sqrt{x^2+9}}\,dx = \frac{-1}{9}\,\frac{\sqrt{x^2+9}}{x} + C$$

FINAL ANSWER: $\int_1^2 \frac{1}{x^2\sqrt{x^2+9}}\,dx = -\frac{1}{9}\left(\frac{\sqrt{13}}{2} - \frac{\sqrt{10}}{1}\right) \approx 0.1510557803$