MATH 122 – THIRD UNIT TEST

Thursday, December 4, 2008.

NAME: SOLUTIONS

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A	B	D	F	H	J
	C	E	G	I	K

Instructions:

1. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
2. Please read the instructions for each individual question carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated. However, if you are asked to find an ***exact*** value of a quantity that involves an integral then you should not use calculator integration for this.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. If you evaluate an improper integral, be sure to use appropriate algebraic and limit notation.
7. Please **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

Problem	Total	Score
1	20	
2	20	
3	30	
4	20	
5	10	
Total	100	

1. **20 Points. SHOW ALL WORK. NO WORK = NO CREDIT.**

Consider the function: $g(x) = \ln(x)$.

(a) **(8 points)** Find the degree 3 Taylor polynomial of $g(x)$ centered at the point $a = 1$.

$$g(x) = \ln(x) \qquad g(a) = g(1) = 0$$
$$g'(x) = \frac{1}{x} \qquad g'(a) = g'(1) = 1$$
$$g''(x) = \frac{-1}{x^2} \qquad g''(a) = g''(1) = -1$$
$$g'''(x) = 2/x^3 \qquad g'''(a) = g'''(1) = 2$$

So the degree 3 Taylor polynomial is:

$$P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

(b) **(4 points)** Use your answer to Part (a) to find an approximate value for ln(1.5). Include at least **8 decimal places** in your answer.

$\ln(1.5) \approx$ 0.41666666

Note: We are not looking for the value of ln(1.5) that your calculator gives. If you write that figure down as your answer you will receive zero credit.

Continued on the next page.

Consider the function: $g(x) = \ln(x)$.

(d) **(8 points)** The Taylor Remainder Theorem states that if $P_N(x)$ is the degree n Taylor polynomial (centered at $x = a$) approximation to the function $g(x)$, then:

$$|g(b) - P_N(b)| \leq \frac{M}{(N+1)!} \cdot |b-a|^{N+1},$$

where M is an appropriate constant. Use the Taylor Remainder Theorem to find a reasonable upper bound for the error that occurs when $P_3(x)$ and $a = 1$ are used to approximate the value of ln(1.5). Be sure to clearly indicate your final answer and show all of your work.

If you express your final answer as a decimal, include at least **8 decimal places.**

To find M, we must estimate the maximum value of $|g''''(x)|$ over the interval $[1, 1.5]$.

$g''''(x) = \frac{-6}{x^4}$. Estimating the maximum on a calculator gives: $M = 6$.

To find the value of N, note that we used the degree 3 Taylor polynomial, so $N = 3$.

We also have $a = 1$ and $b = 1.5$ so:

$$|g(1.5) - P_3(1.5)| \leq \frac{6}{4!}(0.5)^4$$

$$= 0.01562500$$

FINAL ANSWER:

ERROR $\leq$ 0.01562500

SOLUTIONS

2. **20 Points. SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.**

Consider the function: $h(x) = \dfrac{x}{9-x^2}$.

(a) **(8 points)** Write down the Taylor series of the function $h(x)$ about $a = 0$. Express your final answer using Σ notation.

$$h(x) = x \cdot \frac{1}{9 - x^2}$$

$$= x \cdot \frac{1}{9} \cdot \frac{1}{1 - \left(\frac{1}{3}x\right)^2}$$

$$= x \cdot \frac{1}{9} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n} \cdot x^{2n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n+2} \cdot x^{2n+1}$$

FINAL ANSWER:

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n+2} \cdot x^{2n+1}$$

Continued on the next page.

Consider the function: $h(x) = \dfrac{x}{9 - x^2}$.

(c) **(12 points)** Find the **interval** of convergence for the Taylor series of $f(x)$ about $a = 0$. Show your work and clearly indicate your final answer. No work = no credit.

First, use the Ratio Test with $a_n = \left(\frac{1}{3}\right)^{2n+2} x^{2n+1}$ to find the radius of convergence.

$$\frac{a_{n+1}}{a_n} = \frac{x^{2n+3}}{3^{2n+4}} \cdot \frac{3^{2n+2}}{x^{2n+1}} = \frac{x^2}{9}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{9}\left|x^2\right| < 1 \quad \text{so that} \quad |x| < 3.$$

The interval of convergence includes at least $(-3, 3)$. Next we must check convergence at the endpoints.

<u>$x = -3$:</u> Plugging this into the power series gives:

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n+2} \cdot (-1)^{2n+1} \cdot 3^{2n+1} = \sum_{n=0}^{\infty} \frac{-1}{3}.$$

This series diverges by the n^{th} term test for divergence.

<u>$x = 3$:</u> Plugging this into the power series gives:

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n+2} (3)^{2n+1} = \sum_{n=0}^{\infty} \frac{1}{3}.$$

This series diverges by the n^{th} term test for divergence.

So, neither endpoint is included in the interval of convergence.

FINAL ANSWER:

INTERVAL: $(-3, 3)$

3. 30 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.

Determine the convergence or divergence of each of the following series. If you do not justify your answer, you will get zero credit, even if you circle the correct final answer.

In each case, demonstrate that your answer is correct in a step-by-step fashion using *an appropriate convergence test.* Be sure to explicitly state which convergence test you have used and show that it can be used with the series you are working on. Be careful to show all of your work. As the final part of your answer in each part, CIRCLE either CONVERGES or DIVERGES.

(a) **(10 points)** $$\sum_{n=1}^{\infty}\frac{3}{n\cdot(n+3)}$$

JUSTIFICATION: We will use the Comparison Test. Note that $a_n = \dfrac{3}{n(n+3)} = \dfrac{3}{n^2+3n}$.

Initial Guess: When n is large, $\dfrac{3}{n^2+3n} \approx \dfrac{3}{n^2}$.

Now, $\sum_{n=1}^{\infty}\frac{3}{n^2}$ is a p-series with $p = 2 > 1$ so it converges. Because $a_n \approx 3/n^2$ for large n, I guess that $\sum_{n=1}^{\infty}\frac{3}{n^2+3n}$ similarly converges.

Formal Comparison:

$$n^2 + 3n > n^2 \quad (n>0)$$

$$\frac{1}{n^2+3n} < \frac{1}{n^2}$$

$$\frac{3}{n^2+3n} < \frac{3}{n^2}$$

Since $\sum_{n=1}^{\infty} 3/n^2$ (p-series with $p = 2 > 1$) converges and $\dfrac{3}{n^2+3n} < \dfrac{3}{n^2}$, the Comparison test gives that $\sum_{n=1}^{\infty}\frac{3}{n^2+3n}$ also converges.

FINAL ANSWER (CIRCLE ONE): CONVERGES DIVERGES

Demonstrate that your answer is correct in a step-by-step fashion using *an appropriate convergence test*. Be sure to explicitly state which convergence test you have used and show that it can be used with the series you are working on. Be careful to show all of your work. As the final part of your answer in each part, CIRCLE either CONVERGES or DIVERGES.

(b) **(10 points)** $\displaystyle\sum_{n=1}^{\infty} \frac{\pi^n}{3^{2n} \cdot (2n)!}$

JUSTIFICATION: We will use the Ratio Test with

$$a_n = \frac{\pi^n}{3^{2n} \cdot (2n)!}.$$

$$\frac{a_{n+1}}{a_n} = \frac{\pi^{n+1}}{3^{2n+2}\,(2n+2)!} \cdot \frac{3^{2n} \cdot (2n)!}{\pi^n}$$

$$= \frac{\pi}{3^2 \cdot (2n+2)(2n+1)}$$

$$\lim_{n\to\infty} \left| \frac{\pi}{3^2\,(2n+2)(2n+1)} \right| = 0.$$

As the limit is less than 1, the Ratio test gives that the series $\displaystyle\sum_{n=1}^{\infty} \frac{\pi^n}{3^{2n} \cdot (2n)!}$ converges.

FINAL ANSWER (CIRCLE ONE): CONVERGES DIVERGES

Demonstrate that your answer is correct in a step-by-step fashion using *an appropriate convergence test.* Be sure to explicitly state which convergence test you have used and show that it can be used with the series you are working on. Be careful to show all of your work. As the final part of your answer in each part, CIRCLE either CONVERGES or DIVERGES.

(c) **(10 points)** $\sum_{n=2}^{\infty} \frac{4}{n \cdot \sqrt{\ln(n)}}$. You may use the following fact: $\lim_{n \to \infty} \sqrt{\ln(|n|)} = +\infty$.

JUSTIFICATION: We will use the Integral Test with $f(x) = \frac{4}{x \cdot \sqrt{\ln(x)}}$. First we verify that the integral test applies when $x > 2$.

(I) $f(x) > 0$: $f(x) = \frac{+}{+ \cdot \sqrt{+}} = \frac{+}{+} = +$ when $x > 2$

(II) $f'(x) < 0$: $f'(x) = \frac{-4 \cdot \left(\sqrt{\ln(x)} + x \cdot \frac{1}{2} \cdot (\ln(x))^{-1/2} \cdot \frac{1}{x}\right)}{x^2 \cdot \ln(x)}$

$= \frac{(-)(+)}{+} = -$ when $x > 2$

Calculating the improper integral with the u-substitution $u = \ln(x)$ gives:

$$\int_2^\infty \frac{4}{x \cdot \sqrt{\ln(x)}}\,dx = \lim_{a \to \infty} \int_2^a \frac{4}{x\sqrt{\ln(x)}}\,dx$$

$$= \lim_{a \to \infty} \int_{\ln(2)}^{\ln(a)} 4u^{-1/2}\,du$$

$$= \lim_{a \to \infty} \Big[8\sqrt{\ln(x)}\Big]_2^a$$

$$= \lim_{a \to \infty} 8\sqrt{\ln(a)} - 8\sqrt{\ln(2)} = +\infty$$

As the improper integral diverges, the Integral Test gives that the series $\sum_{n=2}^{\infty} 4/n \cdot \sqrt{\ln(n)}$ also diverges.

FINAL ANSWER (CIRCLE ONE): CONVERGES DIVERGES

4. 20 Points. SHOW YOUR WORK.

Consider the alternating series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}.$$

(a) **(10 points)** Does the series converge absolutely, converge conditionally or diverge? Clearly state your answer and use a convergence test to demonstrate that your answer is correct.

Be sure to explicitly state which convergence test you have used and show that it can be used with the series you are working on. Be careful to show all of your work and clearly state your final conclusion.

The series converges absolutely. You can show this using many different tests. Here $a_n = 1/n^5$. We will use the Integral Test with $f(x) = 1/x^5$ but you could use a different test (or observe that $\sum_{n=1}^{\infty} 1/n^5$ is a p-series with $p = 5 > 1$).

Carrying out the integral test we note:

(i) $f(x) = \frac{1}{x^5} = \frac{+}{+} = +$ when $x > 1$.

(ii) $f'(x) = \frac{-5}{x^6} = \frac{-}{+} = -$ when $x > 1$.

Evaluating the improper integral:

$$\int_1^{\infty} \frac{1}{x^5}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x^5}\,dx = \lim_{a\to\infty} \left[\frac{x^{-4}}{-4}\right]_1^a$$

$$= \lim_{a\to\infty} \frac{a^{-4}}{-4} + \frac{1}{4} = \frac{1}{4}$$

As the improper integral converges, the Integral test gives that the series $\sum_{n=1}^{\infty} 1/n^5$ converges and therefore, that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ converges absolutely.

Continued on the next page.

The alternating series from the previous page is:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}.$$

(b) **(6 points)** Let S represent the sum of the series. Suppose that S, the sum of the series, is approximated by the N^{th} partial sum:

$$S_N = \sum_{n=1}^{N}\frac{(-1)^{n+1}}{n^5}.$$

What is the smallest value of N that could be used to approximate S by S_N and an error of less than 0.0001?

The series converges so the Alternating Series Approximation Theorem applies. Using $a_n = 1/n^5$ this means:

$$|S - S_N| < a_{N+1} = \frac{1}{(N+1)^5}$$

To ensure that the error is less than 0.0001, we need to solve the following inequality for N:

$$\frac{1}{(N+1)^5} < 0.0001$$

$$N > \left(\frac{1}{0.0001}\right)^{1/5} - 1 \doteq 5.30957$$

Rounding up, the smallest value of N that will work is $N = 6$.

(c) **(4 points)**. Approximate the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}$ with an error of less than 0.0001. Include at least **8 decimal places** in your answer.

y1 = (-1)^(x+1) / x^5

sum(seq(y1(K), K, 1, 6)) = 0.97208063

5. 10 Points. SHOW ALL WORK AND CLEARLY INDICATE THE FINAL ANSWER.

Consider the infinite series:

$$\sum_{n=1}^{\infty} (\ln(x))^n.$$

For what values of x does this series converge?

We need to determine the interval of convergence for this series. We can begin by using the Ratio Test with $a_n = (\ln(x))^n$. Doing this:

$$\frac{a_{n+1}}{a_n} = \frac{(\ln(x))^{n+1}}{(\ln(x))^n} = \ln(x).$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |\ln(x)| < 1$$ so x must satisfy:

$$-1 < \ln(x) < 1.$$

The solution of $\ln(x) > -1$ is : $x > 1/e$.

The solution of $\ln(x) < 1$ is : $x < e$.

So the solution of $-1 < \ln(x) < 1$ is:

$$1/e < x < e.$$

To check the endpoints, plug $x = 1/e$ into the series. This gives $\sum_{n=1}^{\infty} (-1)^n$ which diverges by the n^{th} term test for divergence. Next, plug $x = e$ into the series. This gives $\sum_{n=1}^{\infty} (1)^{\infty}$ which also diverges by the n^{th} term test for divergence.

FINAL ANSWER:

$$1/e < x < e.$$