MATH 122 – SECOND UNIT TEST

Thursday, October 30, 2008.

NAME: SOLUTIONS

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A	B	D	F	H	J
	C	E	G	I	K

Instructions:

1. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
2. Please read the instructions for each individual question carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated. However, if you are asked to find an ***exact*** value of a quantity that involves an integral then you should not use calculator integration for this.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. If you evaluate an improper integral, be sure to use appropriate algebraic and limit notation.
7. Please **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

Problem	Total	Score
1	20	
2	15	
3	20	
4	12	
5	18	
6	15	
Total	100	

SOLUTIONS

1. 20 Points. SHOW ALL WORK. NO WORK = NO CREDIT.

Solve each of the following initial value problems. Your final answers should not include any unspecified constants. Clearly indicate your final answer, for example by circling it.

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

(a) **(10 points)** $4\cdot y'' - 8\cdot y' + 3\cdot y = 0 \qquad y(0) = 2 \qquad y'(0) = \frac{1}{2}$

The characteristic equation is:

$$4r^2 - 8r + 3 = 0.$$

The roots of the characteristic equation are:

$$r = \frac{8 \pm \sqrt{(-8)^2 - (4)(4)(3)}}{2(4)} = \frac{1}{2}\ ,\ \frac{3}{2}.$$

The solution of the differential equation has the form:

$$y(x) = c_1 e^{1/2\, x} + c_2 e^{3/2\, x}.$$

To determine c_1 and c_2 use $y(0) = 2$ and $y'(0) = \frac{1}{2}$.

$y(0) = 2$: $\quad c_1 + c_2 = 2$

$y'(0) = 1/2$: $\quad \frac{1}{2} c_1 + \frac{3}{2} c_2 = \frac{1}{2}$

Solving these equations gives:

$$c_1 = 5/2 \qquad c_2 = -1/2$$

So that the final answer is:

$$y(x) = 5/2\, e^{1/2\, x} - 1/2\, e^{3/2\, x}.$$

Continued on the next page.

Solve each of the following initial value problems. Your final answers should not include any unspecified constants. Clearly indicate your final answer, for example by circling it.

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(b) **(10 points)** $\dfrac{dy}{dx} - \frac{1}{2}\cdot y = e^{-x}$ $\qquad y(0) = -1$

We will use the technique of Integrating Factors.

To find the integrating factor, note that:

$$p(x) = -1/2$$

$$\int p(x)\,dx = -1/2\,x$$

$$I = e^{\int p(x)\,dx} = e^{-1/2\,x}.$$

Multiplying through by the integrating factor:

$$e^{-1/2\,x}\,\frac{dy}{dx} - \frac{1}{2}e^{-1/2\,x}\cdot y = e^{-1/2\,x}\cdot e^{-x}$$

$$\frac{d}{dx}\left(e^{-1/2\,x}\cdot y\right) = e^{-3/2\,x}.$$

Integrating both sides of the equation:

$$\int \frac{d}{dx}\left(e^{-1/2\,x}\cdot y\right)dx = \int e^{-3/2\,x}\,dx$$

$$e^{-1/2\,x}\cdot y = -\frac{2}{3}e^{-3/2\,x} + C$$

$$y = -2/3\,e^{-x} + Ce^{1/2\,x}$$

To evaluate C use $y(0) = -1$:

$$-1 = -\frac{2}{3} + C \quad \text{so} \quad C = -\frac{1}{3}$$

Final answer:

$$y = -\frac{2}{3}e^{-x} - \frac{1}{3}e^{1/2\,x}$$

SOLUTIONS

2. 15 Points. SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.

Calculate the **exact** volume of each of the three-dimensional solids described in words below. You may use the following trigonometric formulas without having to justify them:

$$\sin^2(\theta) = \tfrac{1}{2}(1 - \cos(2\theta)) \qquad \cos^2(\theta) = \tfrac{1}{2}(1 + \cos(2\theta)).$$

You should not use your calculator on this problem for anything besides arithmetic. In particular, finding antiderivatives or evaluating improper integrals on your calculator is not acceptable.

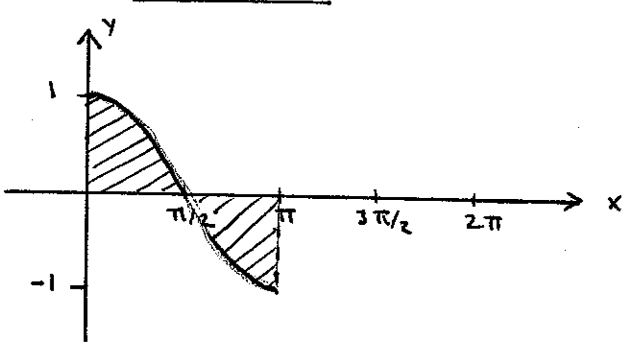
(a) **(10 points)** The base of the solid is the region of the *x*-*y* plane bounded by the following curves:

$$x = 0 \qquad x = \pi \qquad y = 0 \qquad y = \cos(x).$$

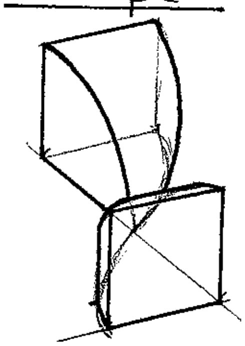
The cross-sections of the solid perpendicular to the *x*-axis are squares.

The following diagram shows the base and overall shape of the solid.

Base



Shape



The volume of an individual slice of the solid with thickness dx is:

$$\text{Slice volume} = \cos^2(x) \cdot dx,$$

so the total volume of the solid is:

$$\text{Volume} = \int_0^\pi \cos^2(x)\,dx = \int_0^\pi \tfrac{1}{2}(1 + \cos(2x))\,dx$$

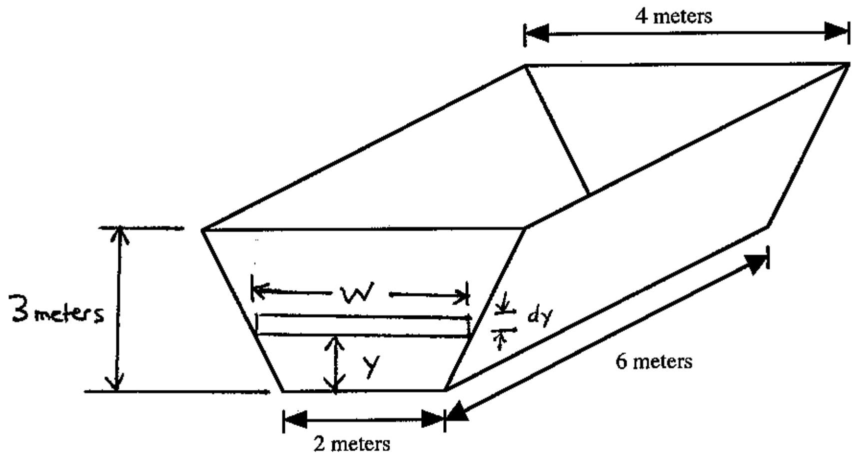
$$= \left[\tfrac{1}{2}x + \tfrac{1}{4}\sin(2x)\right]_0^\pi$$

$$= \pi/2$$

3. 20 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

The object that you will be studying in this problem is a water trough with a trapezoidal end plate as shown in the diagram given below. All dimensions and measurements are in units of meters.



(a) **(10 points)** Suppose that the trough is completely filled with pure water (density = 1000 kg/m^3). Calculate the exact total hydrostatic force exerted on one of the trapezoidal end plates of the trough. Include appropriate units with your answer.

Let $y = 0$ represent the bottom of the water trough.

To determine hydrostatic force, slice the trapezoid into horizontal rectangles (as shown above).

The area of the slice is $w \cdot dy$. To express this entirely in terms of y, find a linear function for w.

w	2	4
y	0	3

$$w = \frac{2}{3} y + 2.$$

So the area of the slice is: $(\tfrac{2}{3}y + 2)\,dy$.

Pressure on the slice is given by: $(9.8)(1000)(3-y)$.

Force = (Pressure)(Area) so the force on the slice is:

$$\text{Force on slice} = (9.8)(1000)(3-y)(\tfrac{2}{3}y+2)\,dy.$$

$$\text{Total force} = \int_0^3 (9.8)(1000)(-\tfrac{2}{3}y^2 + 6)\,dy$$

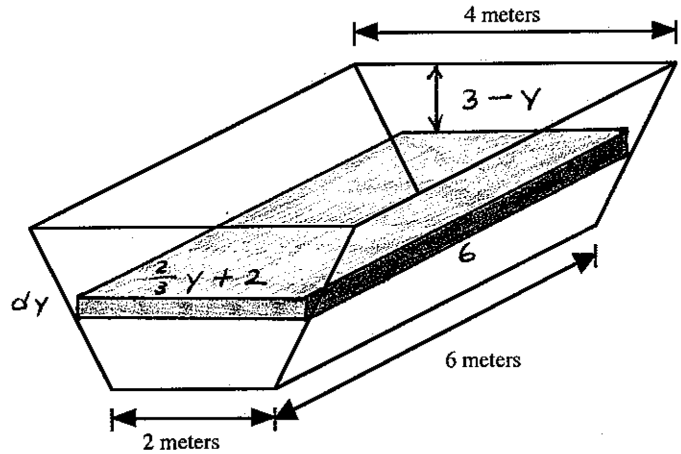
$$= 9800\left[-\tfrac{2}{9}y^3 + 6y\right]_0^3$$

Continued on the next page.

$$= 117{,}600 \text{ N.}$$

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

The object that you will be studying in this problem is a water trough with a trapezoidal end plate as shown in the diagram given below. All dimensions and measurements are in units of meters.



(b) **(10 points)** Suppose that the trough is completely filled with pure water (density = 1000 kg/m^3). Calculate the exact total amount of work that must be done to pump all of the water out of the trough. Include appropriate units with your answer.

Work = (Force) (Distance).

The distance that the slice shown above must move to exit the trough is $3 - y$ meters.

The force is given by g (9.8 ms^{-2}) multiplied by the mass of the slice. This is:

$$\text{Force} = (9.8)(\tfrac{2}{3}y + 2)(6)(1000)\,dy.$$

The work done to remove the slice shown above will be:

$$\text{Work} = (3-y)(9.8)(\tfrac{2}{3}y + 2)(6)(1000)\,dy$$

and the total work will be:

$$\text{Total work} = \int_0^3 (9800)(6)(-\tfrac{2}{3}y^2 + 6)\,dy$$

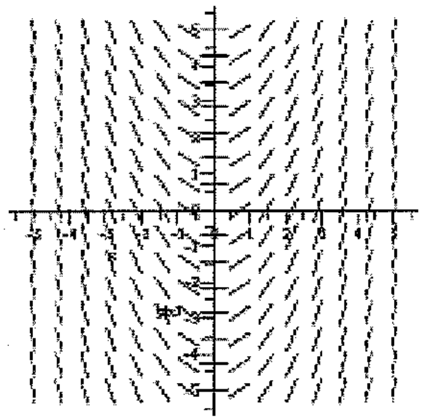
$$= (9800)(6)\left[-\tfrac{2}{9}y^3 + 6y\right]_0^3$$

$$= 705{,}600 \text{ J.}$$

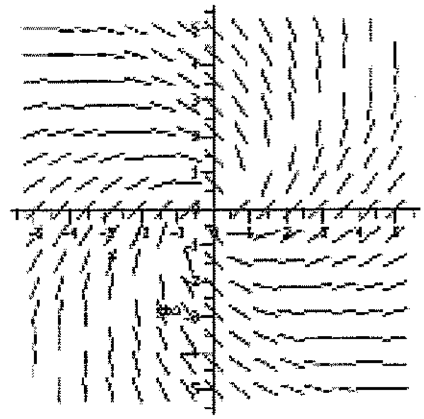
4. 12 Points.

Graphs A, B, C and D show four slope fields.

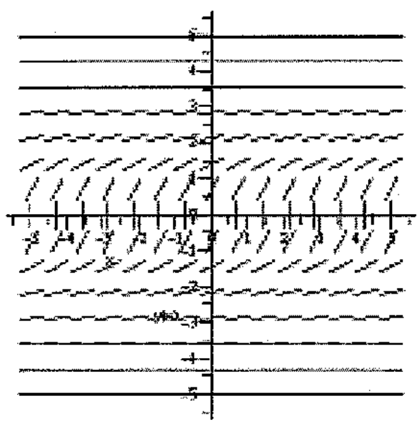
GRAPH A



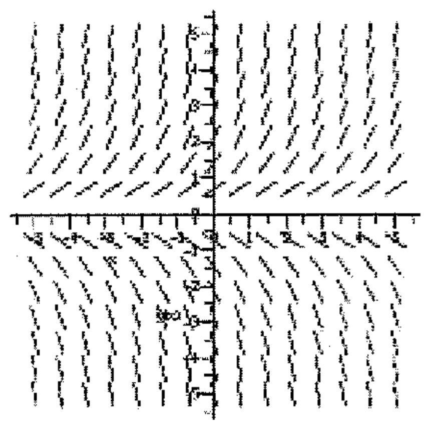
GRAPH B



GRAPH C



GRAPH D



For each of the differential equations listed below, indicate which of the graphs A, B, C, or D does the best job of showing the slope field of that differential equation. If you do not think that any of the slope fields does a good job of showing the slope field of a particular differential equation, write the word **NONE** next to that differential equation.

(a) $\frac{dy}{dx} = x$ **GRAPH:** A

(b) $\frac{dy}{dx} = y$ **GRAPH:** D

(c) $\frac{dy}{dx} = \frac{1}{y^2}$ **GRAPH:** C

(d) $\frac{dy}{dx} = e^{-x}$ **GRAPH:** NONE

(e) $\frac{dy}{dx} = \frac{x-y}{x+y}$ **GRAPH:** NONE

(f) $\frac{dy}{dx} = \frac{x+y}{x-y}$ **GRAPH:** B

5. 18 Points. SHOW ALL WORK. NO WORK = NO CREDIT.

Find the solution to the following initial value problem. Note that your final answer should not contain any unspecified constants. Clearly indicate your final answer.

$$y'' + 5 \cdot y' + 6 \cdot y = x \qquad y(0) = 0 \qquad y'(0) = 0$$

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

Step 1: Homogeneous Equation.

$$y'' + 5y' + 6y = 0$$

Char. Eq$^{\underline{n}}$: $r^2 + 5r + 6 = 0$

$$(r+2)(r+3) = 0$$

Roots: $r = -2, \quad r = -3.$

Homogeneous Solution: $y_h(x) = C_1 e^{-2x} + C_2 e^{-3x}$

Step 2: Particular Solution.

$$\left.\begin{array}{l} N(x) = x \\ N'(x) = 1 \\ N''(x) = 0 \end{array}\right\}$$

Particular solution $y_p(x) = Fx + G$ where F, G are constants.

To determine F and G plug $y_p(x)$ into the nonhomogeneous D.E.

$$y_p''(x) + 5y_p'(x) + 6y_p(x) = x$$

$$0 + 5F + 6Fx + 6G = x$$

Equating coefficients of powers of x: $F = 1/6 \quad G = -5/36.$

Particular solution: $y_p(x) = \frac{1}{6}x - \frac{5}{36}.$

Step 3: Initial Conditions.

$y(0) = 0$: $C_1 + C_2 - 5/36 = 0$

$y'(0) = 0$: $-2C_1 - 3C_2 + 1/6 = 0$

Additional space is provided on the following page if you need it.

Solving for C_1 and C_2: $C_1 = 1/4 \qquad C_2 = -1/9$

FINAL ANSWER: $y(x) = \frac{1}{4}e^{-2x} - \frac{1}{9}e^{-3x} + \frac{1}{6}x - 5/36.$

Find the solution to the following initial value problem. Note that your final answer should not contain any unspecified constants. Clearly indicate your final answer.

$$y'' + 5 \cdot y' + 6 \cdot y = x \qquad\qquad y(0) = 0 \qquad\qquad y'(0) = 0$$

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

FINAL ANSWER: ____________________

SOLUTIONS.

6. 15 Points. CLEARLY INDICATE YOUR FINAL ANSWER.

Endostatin is an experimental drug that might be useful in the fight against cancer. When it is used, a patient begins with no endostatin in their body. The drug is then supplied to the patient by a pump at the steady rate of 25 mg per hour.

The human body eliminates endostatin at a rate that is proportional to the amount of endostatin in the body. The constant of proportionality is 2.7.

The function $y = E(t)$ will represent the amount of endostatin in a patient's body (measured in mg) t hours after the treatment has begun.

(a) **(5 points)** Use the information given above to create a differential equation and an initial condition that describe the rate of change of the amount of endostatin in a patient's body.

$$\frac{dE}{dt} = 25 - 2.7E.$$

$$E(0) = 0.$$

(b) **(6 points)** Find a formula for $E(t)$. Your final answer should contain no unspecified constants.

$$\frac{dE}{dt} = -2.7(E - 25/2.7)$$

$$\int \frac{1}{E - 25/2.7}\, dE = \int -2.7\, dt$$

$$\ln(|E - 25/2.7|) = -2.7t + C$$

$$E = \frac{25}{2.7} + Ae^{-2.7t}, \quad A = \pm e^C$$

To find A use $E(0) = 0$ to get: $A = -25/2.7$.

Final answer: $E(t) = \dfrac{25}{2.7} - \dfrac{25}{2.7} e^{-2.7t}$

(c) **(4 points)** Endostatin is effective when at least 9 mg of the drug are present in a patient's body. How many hours of treatment does it take for the amount of endostatin in a patient's body to reach effective levels?

Solve the following equation for t:

$$9 = \frac{25}{2.7} - \frac{25}{2.7} e^{-2.7t}$$

$$e^{-2.7t} = \frac{2.7}{25}\left(\frac{25}{2.7} - 9\right)$$

$$t = \frac{-1}{2.7} \ln\left(\frac{2.7}{25}\left(\frac{25}{2.7} - 9\right)\right) \approx 1.324278063 \text{ hours}$$