MATH 122 – FIRST UNIT TEST

Thursday, October 2, 2008.

NAME: SOLUTIONS

Sona Akopian	Brian Seguin	Paul McKenney	Oleksii Mostovyi	Jason Rute	Lisa Espig
A	B	D	F	H	J
	C	E	G	I	K

Instructions:

1. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
2. Please read the instructions for each individual question carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated. However, if you are asked to find an ***exact*** value of a quantity that involves an integral then you should not use calculator integration for this.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. If you evaluate an improper integral, be sure to use appropriate algebraic and limit notation.
7. Please **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

Problem	Total	Score
1	16	
2	20	
3	30	
4	20	
5	14	
Total	100	

SOLUTIONS.

1. 16 Points. SHOW ALL WORK. NO WORK = NO CREDIT.

Evaluate each of the indefinite integrals to find the most general antiderivative. In each case, show your work and clearly indicate your final answer. No work, no credit even if your final answer is correct.

You may use the following formulas without having to justify them:

$$\sin^2(\theta) = \tfrac{1}{2}(1-\cos(2\theta)) \qquad \cos^2(\theta) = \tfrac{1}{2}(1+\cos(2\theta)).$$

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

(a) **(8 points)** $\int \tan^3(w)dw$

We will start by using the identity:

$$\tan^2(w) = \sec^2(w) - 1.$$

The integral can then be written as:

$$\int \tan^3(w)\,dw = \int \tan(w)\sec(w)\sec(w)\,dw - \int \tan(w)\,dw.$$

The integral $\int \tan(w)\sec(w)\sec(w)dw$ can be computed by u-substitution using $u = \sec(w)$ and $du = \sec(w)\tan(w)\,dw$.

$$\int \tan(w)\sec(w)\sec(w)\,dw = \int u\,du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\sec^2(w) + C$$

The integral $\int \tan(w)\,dw$ can be rewritten as $\int \frac{\sin(w)}{\cos(w)}\,dw$ and computed via the u-substitution $u = \cos(w)$ and $du = -\sin(w)\,dw$.

$$\int \tan(w)\,dw = \int \frac{\sin(w)}{\cos(w)}\,dw = -\int \frac{1}{u}\,du = -\ln(u) + C = -\ln(|\cos(w)|) + C$$

Putting this all together:

$$\int \tan^3(w)\,dw = \tfrac{1}{2}\sec^2(w) + \ln(|\cos(w)|) + C.$$

Continued on the next page.

Evaluate each of the indefinite integrals to find the most general antiderivative. In each case, show your work and clearly indicate your final answer. No work, no credit even if your final answer is correct.

You may use the following formulas without having to justify them:

$$\sin^2(\theta) = \tfrac{1}{2}(1-\cos(2\theta)) \qquad \cos^2(\theta) = \tfrac{1}{2}(1+\cos(2\theta)).$$

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

(b) **(8 points)** $\int \sin^4(\theta)d\theta$

We will start by using the identity:

$$\sin^2(x) = \tfrac{1}{2}(1 - \cos(2x)).$$

to rewrite the integral as follows

$$\begin{aligned} \int \sin^4(\theta)\,d\theta &= \int \left(\tfrac{1}{2}(1-\cos(2\theta))\right)^2 d\theta \\ &= \tfrac{1}{4}\int (1-\cos(2\theta))^2\,d\theta \\ &= \tfrac{1}{4}\int 1 - 2\cos(2\theta) + \cos^2(2\theta)\,d\theta \\ &= \tfrac{1}{4}\theta - \tfrac{1}{4}\sin(2\theta) + \tfrac{1}{4}\int \cos^2(2\theta)d\theta \end{aligned}$$

To evaluate the last integral, note that:

$$\cos^2(2\theta) = \tfrac{1}{2}(1+\cos(4\theta))$$

so that

$$\begin{aligned} \int \cos^2(2\theta)\,d\theta &= \tfrac{1}{2}\int (1+\cos(4\theta))\,d\theta \\ &= \tfrac{1}{2}\theta + \tfrac{1}{8}\sin(4\theta) + C \end{aligned}$$

Putting all of this together gives:

$$\begin{aligned} \int \sin^4(\theta)\,d\theta &= \tfrac{1}{4}\theta - \tfrac{1}{4}\sin(2\theta) + \tfrac{1}{8}\theta + \tfrac{1}{32}\sin(4\theta) + C \\ &= \tfrac{3}{8}\theta - \tfrac{1}{4}\sin(2\theta) + \tfrac{1}{32}\sin(4\theta) + C \end{aligned}$$

2. 20 Points. SHOW ALL WORK. NO PARTIAL CREDIT WITHOUT WORK.

Determine whether each of the following improper integrals *converges* or *diverges*. Indicate your answer by circling "CONVERGE" or "DIVERGE" next to each integral. If the integral converges, show why (e.g. calculate its value). If the integral diverges, show why.

Limited partial credit may be for correct, appropriate work (if shown) even if your final conclusion is incorrect. You should not use your calculator on this problem for anything besides arithmetic. In particular, finding antiderivatives or evaluating improper integrals on your calculator is not acceptable.

(a) **(10 points)** $\displaystyle\int_2^3 \frac{4}{y-3}\,dy$ **CONVERGE** **DIVERGE.**

We can compute the value of the improper integral directly:

$$\int_2^3 \frac{4}{y-3}\,dy = \lim_{b\to 3^-} \int_2^b \frac{4}{y-3}\,dy$$

$$= \lim_{b\to 3^-} \Big[4\cdot \ln(|y-3|)\Big]_2^3$$

$$= \lim_{b\to 3^-} 4\cdot \ln(|b-3|)$$

$$= -\infty$$

So the improper integral diverges.

Continued on the next page.

Determine whether each of the following improper integrals *converges* or *diverges*. Indicate your answer by circling "CONVERGE" or "DIVERGE" next to each integral. If the integral converges, show why (e.g. calculate its value). If the integral diverges, show why.

Limited partial credit may be for correct, appropriate work (if shown) even if your final conclusion is incorrect. You should not use your calculator on this problem for anything besides arithmetic. In particular, finding antiderivatives or evaluating improper integrals on your calculator is not acceptable.

(b) **(10 points)** $\displaystyle\int_0^\infty \frac{x}{e^{x^2}}\,dx$ **CONVERGE** **DIVERGE.**

We can compute this integral directly using u-substitution.

$$\int \frac{x}{e^{x^2}}\,dx = \int x e^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$$

$$u = -x^2 \qquad \frac{du}{dx} = -2x \qquad dx = \frac{du}{-2x}$$

Computing the improper integral:

$$\int_0^\infty \frac{x}{e^{x^2}}\,dx = \lim_{a\to\infty}\int_0^a \frac{x}{e^{x^2}}\,dx = \lim_{a\to\infty}\left[\frac{-1}{2}e^{-x^2}\right]_0^a = \lim_{a\to\infty}\frac{1}{2} - \frac{1}{2}e^{-a^2} = \frac{1}{2}$$

So the improper integral converges to 1/2.

3. 30 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable. No work = no credit even if your answer is correct. Your answers may contain unspecified constants (such as M, N, B and R).

You may use the following indefinite integral formula without having to justify it:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$$

(a) **(10 points)** Find a formula for the most general antiderivative of: $f(x) = \dfrac{3x^2+4Nx+N^2}{x\cdot(x+N)^2}$ where N is a positive constant.

Use partial fractions to simplify the formula for $f(x)$.

$$\frac{3x^2+4Nx+N^2}{x(x+N)^2} = \frac{A}{x} + \frac{B}{x+N} + \frac{C}{(x+N)^2}$$

Adding the three fractions gives:

$$\frac{3x^2+4Nx+N^2}{x(x+N)^2} = \frac{A(x+N)^2 + Bx(x+N) + Cx}{x(x+N)^2}$$

Concentrating on the denominators, FOILing and equating coefficients of powers of x gives:

$$\begin{array}{llr} \underline{x^2:} & A + B & = 3 \\ \underline{x^1:} & 2NA + BN + C & = 4N \\ \underline{x^0:} & A\cdot N^2 & = N^2 \end{array}$$

So, $A = 1$, $B = 2$ and $C = 0$. This gives:

$$\int f(x)\,dx = \int \frac{1}{x}\,dx + \int \frac{2}{x+N}\,dx$$
$$= \ln(|x|) + 2\cdot\ln(|x+N|) + C$$

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You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable. No work = no credit even if your answer is correct. Your answers may contain unspecified constants (such as M, N, B and R).

You may use the following indefinite integral formula without having to justify it:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C.$$

(b) **(10 points)** Find a formula for the most general antiderivative of: $g(x)=\dfrac{1}{x^2-6Bx+13B^2}$ where B is a positive constant.

The denominator of $g(x)$ does not factor so we will use completing the square.

$$x^2 - 6Bx + 13B^2 = x^2 - 6Bx + 9B^2 - 9B^2 + 13B^2$$
$$= (x-3B)^2 + 4B^2$$

Using the integration formula given above:

$$\int g(x)\,dx = \int \frac{1}{(x-3B)^2+4B^2}\,dx$$
$$= \frac{1}{2B}\arctan\left(\frac{x-3B}{2B}\right) + C$$

Continued on the next page.

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable. No work = no credit even if your answer is correct. Your answers may contain unspecified constants (such as M, N, B and R).

You may use the following indefinite integral formula without having to justify it:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C.$$

(c) **(10 points)** Find a formula for the most general antiderivative of: $j(x) = \dfrac{x^3+Rx^2-2R^2x}{x+2R}$ where R is a positive constant.

We will simplify the formula for $j(x)$ by polynomial long division.

$$\begin{array}{r|l} & x^2 - Rx \\ \hline x+2R & x^3 + Rx^2 - 2R^2x \\ & \underline{x^3 + 2Rx^2} \\ & \quad -Rx^2 - 2R^2x \\ & \quad \underline{-Rx^2 - 2R^2x} \\ & \qquad\qquad 0 \end{array}$$

So:

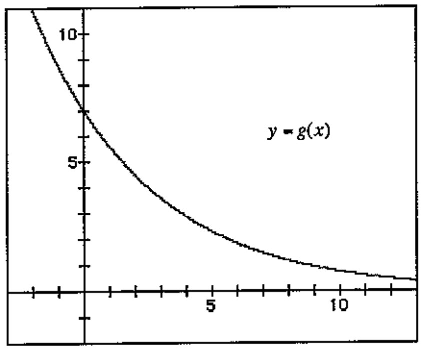
$$\int j(x)\,dx = \int (x^2 - Rx)\,dx$$

$$= \frac{1}{3}x^3 - \frac{1}{2}Rx^2 + C$$

SOLUTIONS

4. 20 Points. SHOW YOUR WORK IN PART (b).

(a) **(10 points)** The function $g(x)$ is defined by the graph shown below.



Arrange the following five quantities from smallest to largest.

(I) $\sum_{k=0}^{3} g(1+k\cdot\frac{3}{2})\cdot\frac{3}{2}$ (II) $\sum_{k=1}^{4} g(1+k\cdot\frac{3}{2})\cdot\frac{3}{2}$ (III) $\int_1^7 g(x)dx$

(IV) $\frac{\sum_{k=0}^{3} g(1+k\cdot\frac{3}{2})\cdot\frac{3}{2}+\sum_{k=1}^{4} g(1+k\cdot\frac{3}{2})\cdot\frac{3}{2}}{2}$ (V) $\sum_{k=0}^{3} g(1+\frac{3}{4}+k\cdot\frac{3}{2})\cdot\frac{3}{2}$

II ≤ V ≤ III ≤ IV ≤ I

SMALLEST **LARGEST**

(b) **(10 points)** How many trapezoids should you use if you wanted to approximate the value of $\int_1^3 \sin(x^2)dx$ using the Trapezoid Rule and with an error of less than 0.01? Show your work.

$a=1$, $b=3$. $f(x) = \sin(x^2)$ $f'(x) = \cos(x^2)\cdot 2x$

$f''(x) = -\sin(x^2)\cdot 4x^2 + 2\cos(x^2)$.

Graphing $|f''(x)|$ between $x=1$ and $x=3$ gives $K = 31.98$

Now, for the trapezoid rule $|\text{Error}| \leq \frac{K\cdot(b-a)^3}{12\cdot N^2}$ so:

$$\frac{(31.98)(3-1)^3}{12\cdot N^2} < 0.01$$

$$N > \sqrt{\frac{(31.98)(3-1)^3}{(12)(0.01)}}$$

$$N > 46.1736$$

To achieve desired accuracy use at least 47 trapezoids.

5. 14 Points. SHOW ALL WORK. NO WORK = NO CREDIT.

Use the technique of Trigonometric substitution to evaluate the definite integral shown below. Show your work and clearly indicate your final answer. No work = no credit even if your answer is correct.

You should not use your calculator on this problem for anything except simple arithmetic. If you need to find any antiderivatives, you should show your work. Finding antiderivatives on a calculator is not acceptable.

$$\int_1^2 \frac{1}{\sqrt{1+z^2}}\,dz$$

You may use the integration formula: $\int \sec(x)dx = \ln(|\sec(x)+\tan(x)|)+C$ without having to justify it.

We will make the substitution: $z = \tan(\theta)$

$dz = \sec^2(\theta)\,d\theta$

$\sqrt{1+z^2} = \sec(\theta)$.

With this, the integral becomes:

$$\int \frac{1}{\sqrt{1+z^2}}\,dz = \int \frac{1}{\sec(\theta)}\cdot\sec^2(\theta)\cdot d\theta$$

$$= \int \sec(\theta)\,d\theta$$

$$= \ln(|\sec(\theta)+\tan(\theta)|)+C$$

Now, $\tan(\theta) = z$ and $\sec(\theta) = \sqrt{1+z^2}$ so:

$$\int \frac{1}{\sqrt{1+z^2}}\,dz = \ln(|z+\sqrt{1+z^2}|)+C$$

Evaluating the definite integral:

$$\int_1^2 \frac{1}{\sqrt{1+z^2}}\,dz = \ln(2+\sqrt{5}) - \ln(1+\sqrt{2})$$

$$\approx 0.5622618882$$

FINAL ANSWER: $\int_1^2 \frac{1}{\sqrt{1+z^2}}\,dz =$ $\ln(2+\sqrt{5}) - \ln(1+\sqrt{2})$