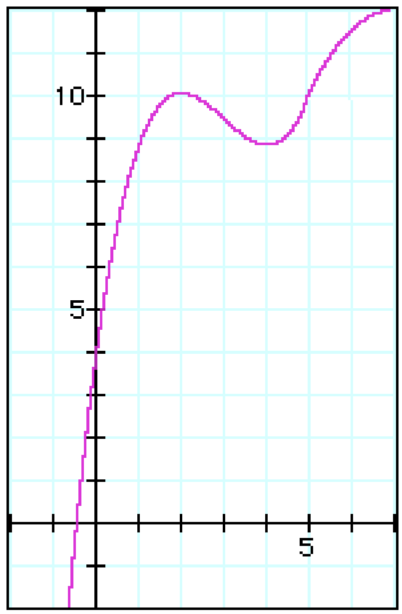
Math 120, Winter 2009. Answers to Unit Test 3 Review Problems – Set B.

Brief Answers. (These answers are provided to give you something to check your answers against. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.)

1. There are many possible answers to this problem. What we are looking for here is that you have the right features (y-intercept, increasing/decreasing and concavity) in the right places. The graph of one function that has the required features is shown below.



2. If the hotel chain takes only 300 chairs then the total revenue will be \$27,000. If the hotel chain takes x chairs (where x is between 300 and 400) then the revenue is given by:

$$R(x) = x\cdot(90 - 0.25\cdot(x - 300)).$$

This problem illustrates the phenomenon that coming up with the function that describes the situation is often the hardest part of an optimization problem. The rationale behind this complicated formula are as follows: First, revenue will be equal to the number of chairs times the price of each chair. Therefore, $R(x) = x\cdot(\text{Price})$. The basic price of each chair is \$90. However, this price is lowered by twenty-five cents for every chair (above 300 chairs) purchased. If x chairs are purchased, but the first 300 do not create a discount, then the number of chairs that create discounts will be: $x - 300$. Each of these chairs creates a twenty-five cent discount, so in units of dollars, the total discount on the chair price will be: $0.25\cdot(x - 300)$. Subtracting this discount from the \$90 basic chair price gives the discounted chair price of: $90 - 0.25\cdot(x - 300)$. Multiplying this price by x, the number of chairs actually sold, gives the revenue, $R(x)$.

Differentiating this function and setting the derivative equal to zero gives $x = 330$. To see this, observe that by FOILing, you can simplify the formula for $R(x)$.

$$R(x) = x \cdot (90 - 0.25 \cdot (x - 300)) = 165 \cdot x - 0.25 \cdot x^2.$$

$$R'(x) = 165 - 0.5 \cdot x$$

Checking the sign of the derivative on each side of $x = 330$ confirms that this is a local maximum.

X	329.9	330	330.1
Derivative	+0.05	0	–0.05

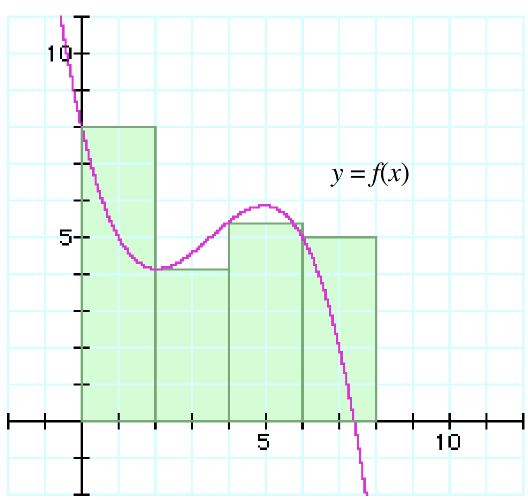
When $x = 330$, the revenue will be $R(330)$ = \$27,225. The one other point to check is the end-point $x = 400$. The revenue when $x = 400$ is: $R(400)$ = \$26,000. Therefore, the greatest revenue that the furniture company can collect is \$27,225 (when 330 chairs are sold) and the lowest is \$26,000 (when 400 chairs are sold). Astonishingly, the company collects more revenue when they sell 300 chairs than when they sell 400 chairs. As the cost of producing 400 chairs will almost certainly be significantly higher than the cost of producing 300 chairs, the company will make much smaller profits from selling 400 chairs, as compared with their profits from selling 300 chairs. That is not very good business practice, so they might want to re-think their pricing strategy.

3.(a) The sum $\sum_{k=0}^{3} f(k \cdot 2) \cdot 2$ represents the approximation to the value of the "true value" of the integral:

$$\int_0^8 f(x) \cdot dx$$

that you would get if you used 4 rectangles (each with a width $\Delta x = 2$) to approximate the area under the curve $y = f(x)$ between $x = 0$ and $x = 8$.

3.(b) The area that would have the same numerical value as the sum $\sum_{k=0}^{3} f(k \cdot 2) \cdot 2$ is shaded in the diagram shown below.



3.(c) The function values needed to approximate the sum $\sum_{k=0}^{3} f(k \cdot 2) \cdot 2$ are given in the table below.

k	$2 \cdot k$	$f(2 \cdot k)$
0	0	8
1	2	4.142857
2	4	5.428571
3	6	5

The numerical value of the sum is:

$$\sum_{k=0}^{3} f(k \cdot 2) \cdot 2 = (8 + 4.142857 + 5.42857 + 5) \cdot 2 = 45.142857.$$

3.(d) Using 100 rectangles to approximate the area between the curve $y = f(x)$ and the x-axis from $x = 0$ to $x = 8$, the relevant commands for a TI-83 calculator are:

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(8-0)/100 [STO⇨] W

Y1 = (-1/7)*(X-1)*(X-3.5)*(X-6) + 5

sum(seq(Y1(0+K*W)*W,K,0,99))
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The numerical value of this approximation of the area under the curve is: 34.76388571.

3.(e) The symbolic expression that will represent the precise value of the area beneath the graph $y = f(x)$ between $x = 0$ and $x = 8$ is:

$$\textit{Precise Value of Area} = \int_0^8 f(x) \cdot dx.$$

3.(f) The equation for the function $f(x)$ is:

$$f(x) = \tfrac{1}{7} \cdot (x-1) \cdot (x-3.5) \cdot (x-6) + 5 = \tfrac{-1}{7} \cdot x^3 + \tfrac{3}{2} \cdot x^2 - \tfrac{61}{14} \cdot x + 8.$$

An anti-derivative, $F(x)$, of $f(x)$ is given by:

$$F(x) = \tfrac{-1}{28} \cdot x^4 + \tfrac{1}{2} \cdot x^3 - \tfrac{61}{28} \cdot x^2 + 8 \cdot x + C.$$

Using this anti-derivative the precise value of the area will be given by:

$$\textit{Precise Value of Area} = \int_0^8 f(x) \cdot dx = F(8) - F(0) \approx .34.28571429$$

4.(a) The critical points of $F(x)$ are the places where $f(x) = 0$. These occur at $x = 3$ and $x = 5$. To classify these:

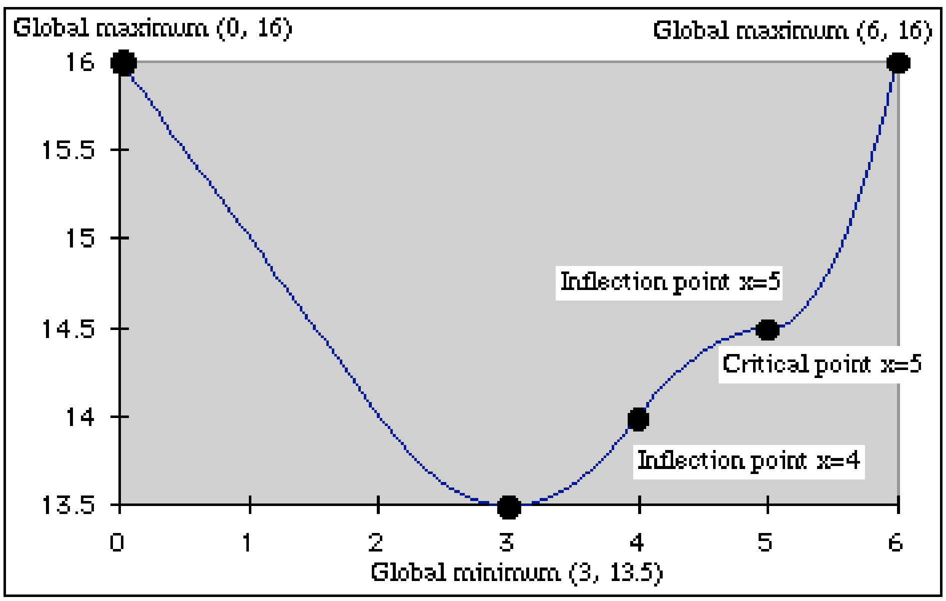
- $x = 3$: The graph of $f(x)$ is negative just before x=3 and positive just after x=3, so by the First Derivative test, x=3 is a local minimum.

- $x = 5$: The graph of $f(x)$ is positive just to the left and just to the right of x=5. This point is neither a maximum nor a minimum.

4.(b) The global minimum is located at x=3. The global maximum occurs at x=0 and x=6.

4.(c) Inflection points are where the graph of $F(x)$ changes concavity. On Figure 4, this is indicated by the graph changing from increasing to decreasing or vice versa. Looking at Figure 4, you can see that this happens at x=4 and x=5. So, $F(x)$ has points of inflection at x=4 and x=5.

4.(d) The graph of $y = F(x)$ with $F(2)$=14 is shown below.



5.(a) In order for $F(x)$ to be an anti-derivative of $f(x)$, the derivative of $F(x)$ should be equal to $f(x)$. That is:

$$F'(x) = f(x).$$

Differentiating the given $F(x) = 2$ yields: $F'(x) = 0$. This is not equal to $f(x) = x^2$, so in this case $F(x)$ is not an anti-derivative of $f(x)$.

5.(b) Differentiating the given function $F(x) = x \cdot \ln(x) - x$ using the Product Rule for derivatives:

$$F'(x) = 1 \cdot \ln(x) - x \cdot \tfrac{1}{x} - 1 = \ln(x) - 1 + 1 = \ln(x).$$

This is equal to $f(x)=\ln(x)$, so in this case $F(x)$ is an anti-derivative for $f(x)$. Using this anti-derivative to evaluate the definite integral $\int_2^{10} f(x)\cdot dx$:

$$\int_2^{10} f(x)\cdot dx = F(10)-F(2)=10\cdot\ln(10)-10-\left[2\cdot\ln(2)-2\right]\approx 13.64\,.$$

5.(c) Differentiating the given function $F(x)=\dfrac{1}{1+x^2}$ using the Quotient Rule for derivatives:

$$F'(x)=\frac{0\cdot\left(1+x^2\right)-2x}{\left(1+x^2\right)^2}=\frac{-2x}{\left(1+x^2\right)^2}.$$

This is equal to $f(x)=\dfrac{-2x}{\left(1+x^2\right)^2}$ so in this case $F(x)$ is an anti-derivative of $f(x)$. Using this anti-derivative to evaluate the definite integral $\int_2^{10} f(x)\cdot dx$:

$$\int_2^{10} f(x)\cdot dx = F(10)-F(2)=\frac{1}{1+10^2}-\frac{1}{1+2^2}\approx -0.19$$

5.(d) Differentiating the given function $F(x)=\ln(\ln(x))$ using the Chain Rule for derivatives:

$$F'(x)=\frac{1}{\left[\ln(x)\right]}\cdot\frac{1}{x}=\frac{1}{x\cdot\ln(x)}.$$

This is not equal to $f(x)=\dfrac{1}{\ln(x)}$ so in this case $F(x)$ is not an anti-derivative of $f(x)$.

5.(e) Differentiating the given function $F(x)=\dfrac{1}{\ln(x)}$ with the Quotient Rule for derivatives gives:

$$F'(x)=\frac{0\cdot\ln(x)-1\cdot\frac{1}{x}}{\left[\ln(x)\right]^2}=\frac{\frac{-1}{x}}{\left[\ln(x)\right]^2}=\frac{-1}{x\cdot\left[\ln(x)\right]^2}.$$

This is equal to $f(x)=\dfrac{-1}{x\cdot\left[\ln(x)\right]^2}$ so in this case $F(x)$ is equal to $f(x)$. Using this anti-derivative to evaluate the definite integral $\int_2^{10} f(x)\cdot dx$:

$$\int_2^{10} f(x)\cdot dx = F(10) - F(2) = \frac{1}{\ln(10)} - \frac{1}{\ln(2)} \approx -1.0084.$$

6.(a) You should use 17 rectangles.

6.(b) Left-hand Riemann sum = 1.915968434.
Right-hand Riemann sum = 2.061109675.

6.(c) The "best estimate" is the average of left and right sums, which is equal to 1.988539055.

6.(d) Midpoint sum = 2.003344759. This is different to the answer from Part (c) of this problem.

7.(a) The table showing the work involved in carrying out Euler's method is given below. According to the figures given in this table, $f(1) \approx 3.99062$.

Current x	Current f(x)	Derivative	Rise	New f(x)
0	1	1	0.25	1.25
0.25	1.25	1.625	0.40625	1.65625
0.50	1.65625	2.99316	0.74829	2.40454
0.75	2.40454	6.344317	1.58607	3.99062

7.(b) The approximation $f(1) \approx 3.99062$ is an underestimate of the true value of $f(1)$. This is because the increasing derivatives in the table from Part (a) show us that the function $f(x)$ is concave up and Euler's method gives an underestimate when the function approximated is concave up.

8.(a) The values of the function $F(x)$ are shown in the table given below.

x	1	2	3	4
F(x)	$8 - \pi / 4$	$17 / 2 - \pi / 4$	$8 - \pi / 4$	$7 - \pi / 4$

9.(a) To find a formula for $\int \frac{2^x}{7+2^x}\cdot dx$ you can:

I) Set $u = 7 + 2^x$.

II) Calculate the derivative of u: $\frac{du}{dx} = \ln(2)\cdot 2^x$.

III) Rearrange to make dx the subject of the equation: $dx = \frac{du}{\ln(2)\cdot 2^x}$.

IV) Substitute into the indefinite integral using the expressions for u and dx: $\int \frac{2^x}{7+2^x}\cdot dx = \int \frac{2^x}{u}\cdot\frac{du}{\ln(2)\cdot 2^x} = \int \frac{1}{\ln(2)}\cdot\frac{1}{u}\cdot du.$

V) Find the anti-derivative, now regarding u as the variable of integration: $\int \frac{1}{\ln(2)} \cdot \frac{1}{u} \cdot du = \frac{1}{\ln(2)} \cdot \ln(u) + C$.

VI) Convert the anti-derivative back to variable x: $\int \frac{2^x}{7+2^x} \cdot dx = \frac{1}{\ln(2)} \cdot \ln\left(7 + 2^x\right) + C$.

9.(b) To find a formula for $\int (4x-6) \cdot \left(x^2 - 3x + 8\right)^2 \cdot dx$ you can:

I) Set $u = x^2 - 3x + 8$.

II) Calculate the derivative of u: $\dfrac{du}{dx} = 2x - 3$.

III) Rearrange to make dx the subject of the equation: $dx = \dfrac{du}{2x - 3}$.

IV) Substitute into the indefinite integral using the expressions for u and dx: $\int (4x-6) \cdot \left(x^2 - 3x + 8\right)^2 \cdot dx = \int (4x-6) \cdot u^2 \cdot \frac{du}{2x-3} = \int 2u^2 \cdot du$.

V) Find the anti-derivative, now regarding u as the variable of integration: $\int 2u^2 \cdot du = \frac{2}{3} \cdot u^3 + C$.

VI) Convert the anti-derivative back to variable x:
$\int (4x-6) \cdot \left(x^2 - 3x + 8\right)^2 \cdot dx = \frac{2}{3} \cdot \left(x^2 - 3x + 8\right)^3 + C$.

9.(c) To find a formula for $\int \dfrac{x}{\left(1 + x^2\right)^2} \cdot dx$ you can:

I) Set $u = 1 + x^2$.

II) Calculate the derivative of u: $\dfrac{du}{dx} = 2x$.

III) Rearrange to make dx the subject of the equation: $dx = \dfrac{du}{2x}$.

IV) Substitute into the indefinite integral using the expressions for u and dx: $\int \frac{x}{\left(1+x^2\right)^2} \cdot dx = \int \frac{x}{u^2} \cdot \frac{du}{2x} = \int \frac{1}{2} \cdot \frac{1}{u^2} \cdot du = \int \frac{1}{2} \cdot u^{-2} \cdot du$.

V) Find the anti-derivative, now regarding u as the variable of integration: $\int \frac{1}{2} \cdot u^{-2} \cdot du = \dfrac{1}{2} \cdot \dfrac{u^{-1}}{-1} + C = \dfrac{-1}{2u} + C$.

VI) Convert the anti-derivative back to variable x: $\int \frac{x}{\left(1+x^2\right)^2} \cdot dx = \dfrac{-1}{2 \cdot \left(1 + x^2\right)} + C$.

9.(d) To find a formula for $\int \dfrac{9x^2}{\sqrt{1 - x^3}} \cdot dx$ you can:

I) Set $u = 1 - x^3$.

II) Calculate the derivative of u: $\dfrac{du}{dx} = -3x^2$.

III) Rearrange to make dx the subject of the equation: $dx = \frac{du}{-3x^2}$.

IV) Substitute into the indefinite integral using the expressions for u and dx: $\int \frac{9x^2}{\sqrt{1-x^3}} \cdot dx = \int \frac{9x^2}{\sqrt{u}} \cdot \frac{du}{-3x^2} = \int -3 \cdot \frac{1}{\sqrt{u}} \cdot du = \int -3 \cdot u^{-\frac{1}{2}} \cdot du$.

V) Find the anti-derivative, now regarding u as the variable of integration: $\int -3 \cdot u^{-\frac{1}{2}} \cdot du = -3 \cdot \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = -6 \cdot \sqrt{u} + C$.

VI) Convert the anti-derivative back to variable x: $\int \frac{9x^2}{\sqrt{1-x^3}} \cdot dx = -6 \cdot \sqrt{1-x^3} + C$.

9.(e) To find a formula for $\int \frac{14x}{7x^2+7} \cdot dx$ you can:

I) Set $u = 7x^2 + 7$.

II) Calculate the derivative of u: $\frac{du}{dx} = 14x$.

III) Rearrange to make dx the subject of the equation: $dx = \frac{du}{14x}$.

IV) Substitute into the indefinite integral using the expressions for u and dx: $\int \frac{14x}{7x^2+7} \cdot dx = \int \frac{14x}{u} \cdot \frac{du}{14x} = \int \frac{1}{u} \cdot du$.

V) Find the anti-derivative, now regarding u as the variable of integration: $\int \frac{1}{u} \cdot du = \ln(u) + C$.

VI) Convert the anti-derivative back to variable x: $\int \frac{14x}{7x^2+7} \cdot dx = \ln(7x^2+7) + C$.

9.(f) To find a formula for $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \cdot dx$ you can:

I) Set $u = \sqrt{x}$.

II) Calculate the derivative of u: $\frac{du}{dx} = \frac{1}{2 \cdot \sqrt{x}}$.

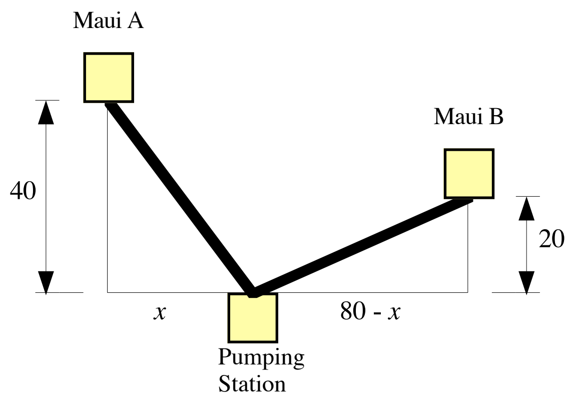
III) Rearrange to make dx the subject of the equation: $dx = 2 \cdot \sqrt{x} \cdot du$.

IV) Substitute into the indefinite integral using the expressions for u and dx: $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \cdot dx = \int \frac{e^u}{\sqrt{x}} \cdot 2 \cdot \sqrt{x} \cdot du = \int 2 \cdot e^u \cdot du$.

V) Find the anti-derivative, now regarding u as the variable of integration: $\int 2 \cdot e^u \cdot du = 2 \cdot e^u + C$.

VI) Convert the anti-derivative back to variable x: $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \cdot dx = 2 \cdot e^{\sqrt{x}} + C$.

10. The most cost-effective solution will probably be the one that involves the shortest possible pipelines. Therefore, the quantity that you should try to minimize here is the total length of the pipeline. If the variable x represents the distance (along the coastline) between the Maui A platform and the pumping station, then the situation can be represented as:



Using the Pythagorean Theorem the total length of the pipeline required to connect both platforms to the pumping station can be expressed as a function of x:

$$L(x) = \sqrt{40^2 + x^2} + \sqrt{20^2 + (80 - x)^2} .$$

Differentiating with respect to x:

$$L'(x) = \frac{x}{\sqrt{40^2 + x^2}} - \frac{(80 - x)}{\sqrt{20^2 + (80 - x)^2}} .$$

Setting the derivative equal to zero and solving for x gives: $x = 160/3$ km. Therefore, in order minimize the total length of the two pipelines, the pumping station should be built 160/3 km from the Maui A platform (distance measured along the Taranaki coast).