Math 120, Winter 2009. Answers to Unit Test 3 Review Problems – Set A.

Brief Answers. (These answers are provided to give you something to check your answers against. Remember than on an exam, you will have to provide evidence to support your answers and you will have to explain your reasoning when you are asked to.)

1.(a) The domain of $g(x)$ is the interval [-4, 4].

1.(b) The x-coordinates of the points where the derivative of $f(x)$ is equal to zero are: $x = -3, -2, -1, 0, 1, 2, 3$. The derivative of $f(x)$ is not defined at either of the points $x = -4$ or $x = 4$ and so cannot be equal to zero at these points.

1.(c) $g'(x) = 2\cdot f(x)\cdot f'(x) - 2\cdot f'(x) = 2\cdot f'(x)\cdot\left[f(x)-1\right]$.

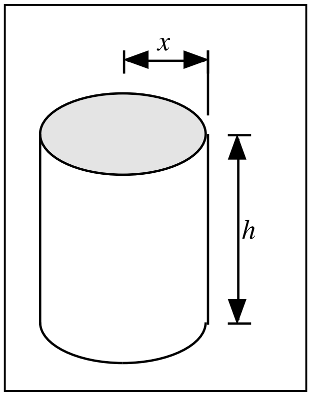
1.(d) Based on the answer to part (c), $g'(x) = 0$ when either $f'(x) = 0$ or when $f(x) = 1$. The points at which $g'(x) = 0$ are: $x = -3.5, -3, -2.5, -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2, 2.5, 3$ and 3.5.

1.(e) Classifying the critical points from Part (d) is not all that easy, but you can do it with just the information contained in the graph of $y = f(x)$ and the equation for $g'(x)$. The method for classifying each critical points (as a maximum, a minimum or neither) that is used here is to look at the sign of $g'(x)$ just to the left and just to the right of the critical point. For example, for the critical point located at $x = -3.5$:

> **Just to the left of $x = -3.5$:** $f(x)$ is slightly larger than 1, so $[f(x) - 1]$ is positive. However, the graph of $y = f(x)$ is decreasing so $f'(x) < 0$. Therefore $g'(x)$ is equal to a positive times a negative and $g'(x)$ is negative just to the left of $x = -3.5$.
>
> **Just to the right of $x = -3.5$:** $f(x)$ is slightly smaller than 1, so $[f(x) - 1]$ is negative. The graph of $y = f(x)$ is still decreasing so $f'(x) < 0$. Therefore $g'(x)$ is equal to a negative times a negative, and so $g'(x)$ is positive just to the right of -3.5.

Therefore, $g(x)$ has a local minimum at $x = -3.5$. The classification of each critical point is given in the table below.

Point	Classification	Point	Classification
$x = -3.5$	Local minimum	$x = 0.5$	Local minimum
$x = -3$	Local maximum	$x = 1$	Local maximum
$x = -2.5$	Local minimum	$x = 1.5$	Local minimum
$x = -2$	Local maximum	$x = 2$	Local maximum
$x = -1.5$	Local minimum	$x = 2.5$	Local minimum
$x = -1$	Local maximum	$x = 3$	Local maximum
$x = -0.5$	Local minimum	$x = 3.5$	Local minimum
$x = 0$	Local maximum		



2. The short answer is: Radius = 3.4139 cm. Height = 13.6556 cm.

Probably the trickiest part of this problem (as is the case with most optimization problems) is setting up the function to maximize or minimize. In this case, the company wants to minimize the cost of its packaging, so we will try to come up with a function that will give the cost of the packaging as a function of either the height (h) or the radius (x) of the cylindrical package.

Referring to the diagram shown above, the total area of the (two) ends of the oil can is: $2\cdot\pi\cdot x^2$. The total area of the sides of the oil can is: $2\cdot\pi\cdot x\cdot h$. Therefore, the cost (C) of the packaging (in cents) will be:

$$C = 4\cdot\pi\cdot x^2 + 2\cdot\pi\cdot x\cdot h.$$

This has too many variables in it to differentiate, so we will use the fact that the volume of the can has to be 500ml to create an equation to eliminate one of the variables (I will eliminate h but you could eliminate x if you really wanted to and still get the right answer). The volume of the cylinder is: $\pi\cdot x^2\cdot h$. Setting this equal to 500 and rearranging to make h the subject of the equation gives the following.

$$h = \frac{500}{\pi\cdot x^2}.$$

Substituting this into the expression for the cost gives:

$$C = 4\cdot\pi\cdot x^2 + 2\cdot\pi\cdot x\cdot\frac{500}{\pi\cdot x^2} = 4\cdot\pi\cdot x^2 + \frac{1000}{x}.$$

Differentiating gives:

$$C' = 8\cdot\pi\cdot x - \frac{1000}{x^2}.$$

Setting this derivative equal to zero and then solving for x gives that the cost function will have a critical point when:

$$8\cdot\pi\cdot x = \frac{1000}{x^2}$$

$$x^3 = \frac{1000}{8\cdot\pi}$$

$$x = \left(\frac{1000}{8\cdot\pi}\right)^{\frac{1}{3}} \approx 3.414.$$

You can check that this is a minimum by evaluating the derivative slightly to the left and slightly to the right of the point and observe the negative to positive pattern of the derivative, which confirms that the critical point is, indeed, a minimum.

X	3.4	3.414	3.5
Derivative	−1.054	0	6.332

3.(a) The critical points of the original function, $f(x)$, are the points where the first derivative, $f'(x)$ is equal to zero. The critical points of $f(x)$ are located at the same x-coordinates as the x-intercepts of the derivative, $f'(x)$. If you inspect the graph, then you can see that these occur at $x = 1$ and $x = 3$.

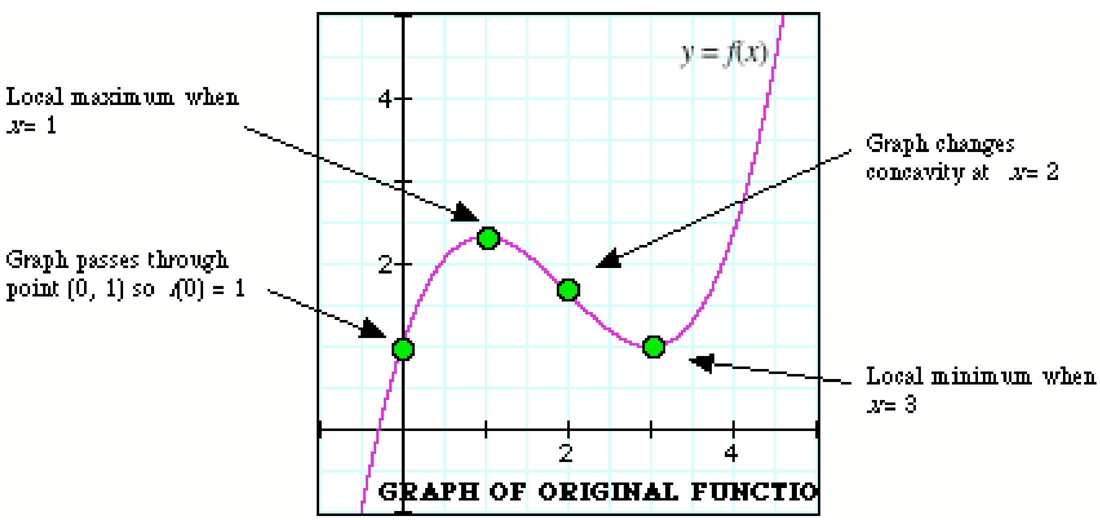
3.(b) **Classifying the critical points of $f(x)$:** If you look at the graph of the derivative, you can see that slightly to the left of $x = 1$, the derivative is positive and slightly to the right of $x = 1$, the derivative is negative. When the first derivative goes from positive to negative, the critical point is a local maximum.

Similarly, from the graph of the derivative, if you look slightly to the left of $x = 3$ then you can see that the derivative is negative whereas slightly to the right of $x = 3$ the derivative is positive. When the first derivative goes from negative to positive the critical point is a local minimum.

3.(c) The definition of a point of inflection is that this is where the graph of $f(x)$ changes concavity. You can locate this point from the derivative graph by following the following reasoning:

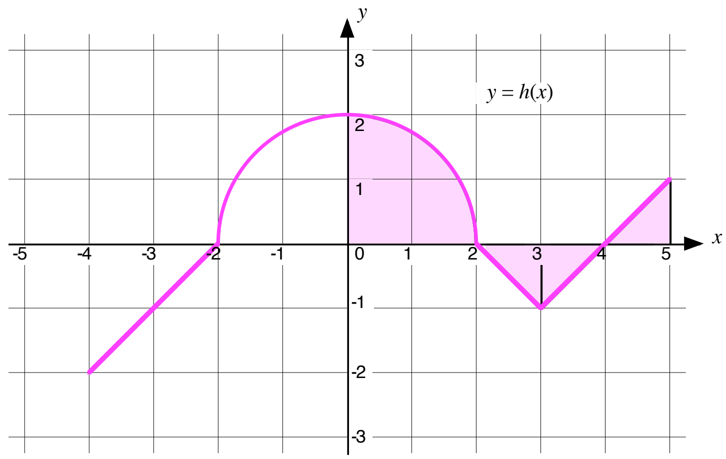
When the derivative is decreasing, the graph of the original function $f(x)$ is concave down. When the derivative is increasing, the graph of the original function $f(x)$ is concave up. The point(s) of inflection of $f(x)$ will occur at any point where the derivative graph goes from increasing to decreasing or vice versa. If you look at the derivative graph, you will be able to see that the only point where this occurs is at $x = 2$.

3.(d) A possible[1] graph of the original function $y = f(x)$ is shown below.

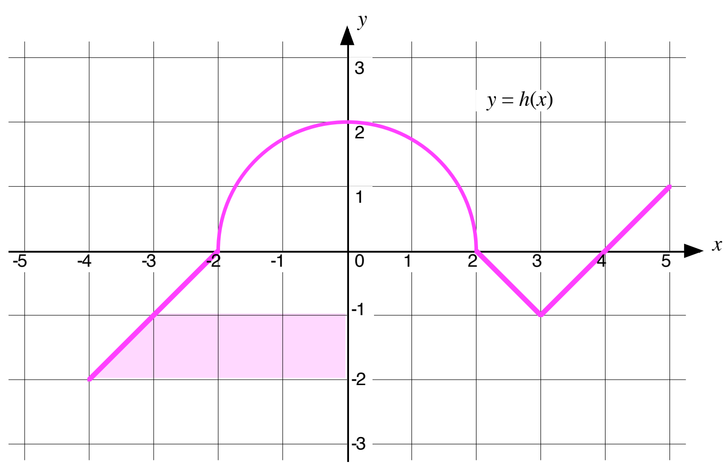


[1] There are many possible answers here, although correct answers will quite closely resemble the graph given. What we were looking for was that you included the point (0, 1) on the graph, had the critical points and inflection point in the right places, indicated the locations of these special points and had a plausible shape for the graph overall.

4.(a) The area is shown in the diagram given below.



4.(b) The area is shown in the diagram given below. Note that instead of the usual "*dx*" this integral had "*dy*' in it, indicating that the area was going to be between the curve and the y-axis. When this is the case, the limits of integration given on the integral will refer to *y*-values rather than *x*-values.



4.(c) It is actually quite difficult to calculate this area using anti-derivatives. The equation for the semi-circular portion of the graph of $y = h(x)$ is given by the equation:

$$y = \sqrt{4 - x^2} ,$$

so the definite integral that you would have to evaluate in order to get the area using calculus is:

$$\int_0^2 \sqrt{4 - x^2} \cdot dx .$$

In Math 1b this integral would be evaluated using an integration technique called "trigonometric substitution." In Math Xb, you are not expected to know how to do this, so there must be another way to work out the numerical value of $\int_0^2 h(x)\cdot dx$. The key here is to realize that the area in question is one quarter of the area of a circle, the radius of the circle being equal to two. Therefore:

$$\int_0^2 h(x)\cdot dx = \frac{\pi\cdot r^2}{4} = \frac{\pi\cdot 2^2}{4} = \pi.$$

4.(d) The symbolic expression $\int_3^5 h(x)\cdot dx$ refers to the triangular area beneath the x-axis between $x = 3$ and $x = 4$, together with the triangular area above the x-axis between $x = 4$ and $x = 5$. As the triangular area between $x = 3$ and $x = 5$ lies below the x-axis, it should be regarded as negative. Therefore:

$$\int_3^5 h(x)\cdot dx = \tfrac{-1}{2} + \tfrac{1}{2} = 0.$$

5.(a) To verify that $F(x)$ is an anti-derivative of $f(x)$, you take the derivative of $F(x)$ and check to make sure that this derivative is equal to $f(x)$. Doing this:

$$F'(x) = -e^{-x} + (-x)\cdot e^{-x}\cdot(-1) - e^{-x}\cdot(-1) = -e^{-x} + x\cdot e^{-x} + e^{-x} = x\cdot e^{-x} = f(x).$$

5.(b) The anti-derivative $F(x)$ can be used to evaluate the definite integral. To do this, you plug the limits of integration ($x = 2$ and $x = 0$) into the formula for $F(x)$. Doing this gives:

$$\int_0^2 f(x)\cdot dx = F(2) - F(0) = 0.59399.$$

5.(c) The anti-derivative can be thought of as the original function that your formula was obtained from by the process of differentiation. The formula:

$$f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

looks a lot like the PRODUCT RULE for derivatives. So, the original function that you must have started with in order to get $f'(x)\cdot g(x) + f(x)\cdot g'(x)$ by differentiating must have been:

$$H(x) = f(x)\cdot g(x) + C.$$

5.(d) In much the same way as in Part (b), this anti-derivative can be used to evaluate the definite integral. To get the values of $f(x)$ you just plug the appropriate x-value into the formula for $f(x)$, and to get the numerical values of $g(x)$, you use the specific values provided in the problem.

$$\int_2^5 h(x)\cdot dx = f(5)\cdot g(5) - f(2)\cdot g(2) = -1.725$$

6.(a) The units of the area under the curve are *billions of dollars*. The units of an area under a graph can be determined by multiplying the units of the quantity represented on the vertical axis by the units of the quantity represented on the horizontal axis. In this case:

$$\begin{aligned} \textit{Units of Area} &= (\textit{Units of vertical axis})\times(\textit{Units of horizontal axis}) \\ &= \left(\frac{\text{Billions of dollars}}{\text{Years}}\right)\times(\text{Years}) \\ &= \text{Billions of dollars} \end{aligned}$$

6.(b) The interpretation of the area under the curve between $T = 1$ and $T = 12$ is that this is the *net change in Argentina's gross domestic product (in units of billions of dollars) between 1991 and 2002*.

6.(c) Between 1999 and the present day ($T = 9$ to $T = 12$) the Argentine GDP has been *decreasing*. You can tell this because the given graph is a graph of the *rate of change of GDP*. This graph is negative from $T = 9$ to $T = 12$ indicating that the rate of change of Argentina's GDP has been negative between $T = 9$ and $T = 12$. When the rate of change is negative, the function (in this case Argentina's GDP) is decreasing.

6.(d) The equation given for the rate of change $r(T)$ was:

$$r(T) = -0.15\cdot T^3 + 2.5\cdot T^2 - 17.7\cdot T + 59.1.$$

Using the short-cut anti-differentiation rule for power functions on a term-by-term basis gives the following equation for the anti-derivative $R(T)$:

$$R(T) = \tfrac{-0.15}{4}\cdot T^4 + \tfrac{2.5}{3}\cdot T^3 - \tfrac{17.7}{2}\cdot T^2 + 59.1\cdot T + C.$$

6.(e) The total change in Argentina's GDP between 1991 and 2002 is given by the numerical value of the definite integral:

$$\textit{Total Change in Argentina's GDP} = \int_1^{12} r(T)\cdot dT = R(12) - R(1) = 46.15 \text{ \$billion}$$

7.(a) The values of the function $F(x)$ are shown in the table given below.

x	1	2	3	4	5
F(x)	1002	3002	4002	5002	5502

7.(b) The units of the area under the graph are: lemmings.

7.(c) x is the number of days that lemming observations have been carried out. $f(x)$ is the rate at which new lemmings are observed during day 'x.' $F(x)$ is the total number of lemmings observed (plus 2) since the beginning of lemming observations.

8.(a) Left hand sum = 6.976174136.

8.(b) Right hand sum = 6.994482433.

8.(c) Midpoint sum = 6.988646849.

8.(d) Trapezoid sum = 6.9885902.

9.(a) Profit = 256q - 1000 - q^3.

9.(b) Profit is maximized at a production level of 9 consoles.

9.(c) The company should keep the price at $256.

10. The general outline for how to use the technique of u-substitution to find equations for anti-derivatives is as follows.

- **a.** Inspect the indefinite integral and decide what u should be.
- **b.** Calculate the derivative $\frac{du}{dx}$.
- **c.** Rearrange the equation for $\frac{du}{dx}$ to make dx the subject of the equation.
- **d.** Using the equations that you have for u and dx, eliminate all traces of x and dx from the indefinite integral.
- **e.** Once the indefinite integral has been expressed entirely in terms of u and du, find the anti-derivative regarding u is the variable.
- **f.** Convert the equation for the anti-derivative back to the variable x by using the equation for u.

10.(a) The indefinite integral is: $\int 20\cdot\left(x^2+x+1\right)^{19}\cdot\left[2x+1\right]\cdot dx$.

i. $u = x^2 + x + 1$

ii. $\frac{du}{dx} = 2x+1$

iii. $dx = \frac{du}{2x+1}$

iv. $\int 20\cdot\left(x^2+x+1\right)^{19}\cdot\left[2x+1\right]\cdot dx = \int 20\cdot u^{19}\cdot du$

v. $\int 20\cdot u^{19}\cdot du = u^{20} + C$

vi. $u^{20} + C = (x^2 + x + 1)^{20} + \mathrm{C}$.

10.(b) The indefinite integral is: $\int \frac{1}{2}\cdot\left(x+\ln(x)\right)^{\frac{-1}{2}}\cdot\left[1+\frac{1}{x}\right]\cdot dx$.

i. $u = x + \ln(x)$

ii. $\frac{du}{dx} = x + \frac{1}{x}$

iii. $dx = \dfrac{du}{1+\frac{1}{x}}$

iv. $\int \frac{1}{2} \cdot \left(x + \ln(x)\right)^{\frac{-1}{2}} \cdot \left[1 + \frac{1}{x}\right] \cdot dx = \int \frac{1}{2} \cdot u^{\frac{-1}{2}} \cdot du$

v. $\int \frac{1}{2} \cdot u^{\frac{-1}{2}} \cdot du = u^{\frac{1}{2}} + C$

vi. $u^{\frac{1}{2}} + C = \sqrt{x + \ln(x)} + C$

10.(c) The indefinite integral is: $\displaystyle\int \frac{6x^2+4}{x^3+2x+10} \cdot dx$.

i. $u = x^3 + 2x + 10$

ii. $\dfrac{du}{dx} = 3x^2 + 2$

iii. $dx = \dfrac{du}{3x^2+2}$

iv. $\displaystyle\int \frac{6x^2+4}{x^3+2x+10} \cdot dx = \int \frac{2}{u} \cdot du$

v. $\displaystyle\int \frac{2}{u} \cdot du = 2 \cdot \ln(u) + C$

vi. $2 \cdot \ln(u) + C = 2 \cdot \ln\left(x^3 + 2x + 10\right) + C$

10.(d) The indefinite integral is: $\displaystyle\int \frac{e^{\sqrt{x}}}{2 \cdot \sqrt{x}} \cdot dx$.

i. $u = \sqrt{x}$

ii. $\dfrac{du}{dx} = \dfrac{1}{2 \cdot \sqrt{x}}$

iii. $dx = 2 \cdot \sqrt{x} \cdot du$

iv. $\displaystyle\int \frac{e^{\sqrt{x}}}{2 \cdot \sqrt{x}} \cdot dx = \int e^u \cdot du$

v. $\int e^u \cdot du = e^u + C$

vi. $e^u + C = e^{\sqrt{x}} + C$