Outline

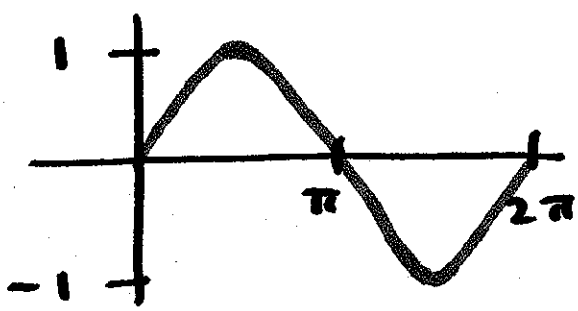
1. Squeeze lemma.
2. Limits as $x \to \pm\infty$.
3. Limits when $f(x)$ becomes infinite.
4. Vertical and horizontal asymptotes.

1. Squeeze Lemma

Example

Calculate $\displaystyle\lim_{x\to 0} x \cdot \sin\left(\frac{1}{x}\right)$.

Solution



$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

Multiply through by x:

$$\underbrace{-x}_{f(x)=-x} \le \underbrace{x\cdot\sin\left(\frac{1}{x}\right)}_{g(x)=x\cdot\sin\left(\frac{1}{x}\right)} \le \underbrace{x}_{h(x)=x.}$$

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} x = 0.$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} -x = 0.$$

By the Squeeze Lemma:

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0.$$

2. Limits as $x \to \pm\infty$.

- The notation:

$$\lim_{x \to \infty} \underset{\text{function}}{f(x)} = \underset{\text{fixed number}}{L}$$

means that as x is allowed to grow larger and larger, the y-value of $f(x)$ gets closer and closer to the y-value L.

Example

What are the values of:

(a) $\lim_{x\to\infty} \dfrac{k}{x^n}$ (b) $\lim_{x\to\infty} kx^n$

where $k > 0$ and $n > 0$.

Solution

(a) k = fixed number.

$$\lim_{x\to\infty} \frac{k}{x^n} = 0$$

The denominator gets really big numerator stays the same.

(b) $\lim_{x\to\infty} k \cdot x^n = \infty.$

x^n gets really, really big given enough time to grow, eventually overwhelming k, no matter how small k is.

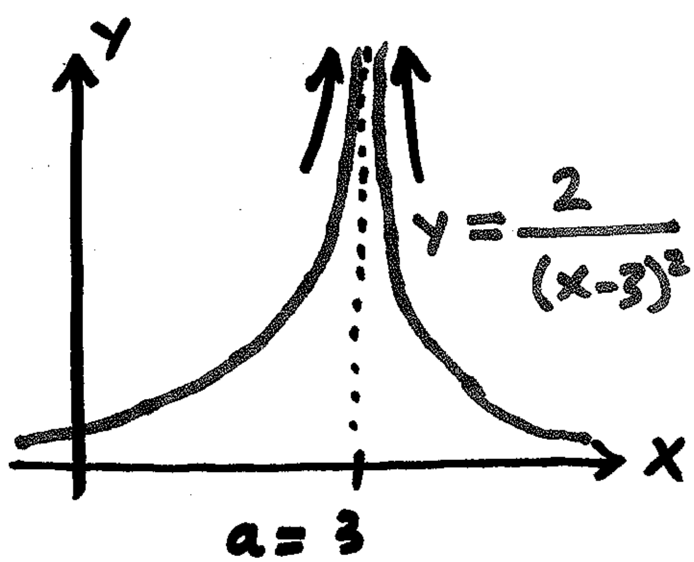
3. Limits When f(x) Becomes Infinite

- The notation: $\lim\limits_{x \to a} f(x) = \infty$

means that as x gets close to the x-value 'a' (approaching from either the left or the right of 'a'), then the y-value produced by f(x) gets larger and larger without any upper bound on its size.

e.g. $f(x) = \dfrac{2}{(x-3)^2}$

$$\lim_{x \to 3} \frac{2}{(x-3)^2} = +\infty.$$

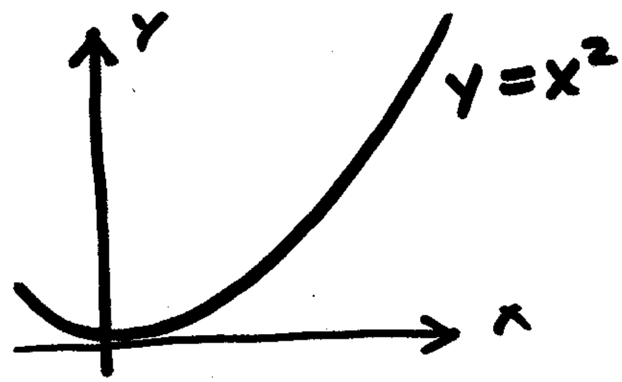


- The notation: $\lim_{x \to \infty} f(x) = +\infty$

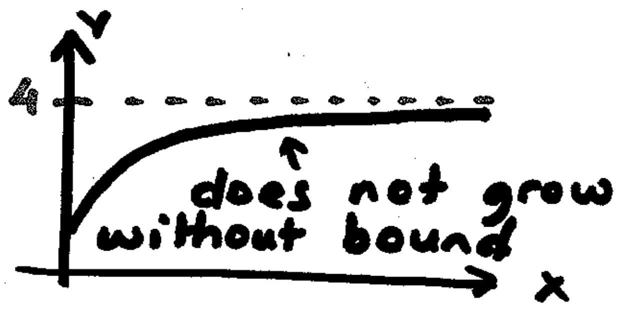
means that as x gets larger and larger, f(x) gets larger and larger without any upper limit on how big it gets.

e.g. $f(x) = x^2$

$\lim_{x \to \infty} f(x) = +\infty$



e.g. $f(x) = \dfrac{4}{1 + e^{-x}}$



$e \approx 2.718$

$\lim_{x \to \infty} f(x) = 4$ (not $+\infty$).

Example

Calculate: $\lim_{x \to \infty} \dfrac{3x^2 - 2x + 1}{7x^2 + x + 14}$

Solution

when dealing with a rational function:

1. Find the term in the denominator with the biggest power of x.
2. Divide everything by that.
3. Take limits as $x \to \infty$.

① $7x^2$

② $$\frac{\frac{3x^2}{7x^2} - \frac{2x}{7x^2} + \frac{1}{7x^2}}{\frac{7x^2}{7x^2} + \frac{x}{7x^2} + \frac{14}{7x^2}} = \frac{\frac{3}{7} - \frac{2}{7x} + \frac{1}{7x^2}}{1 + \frac{1}{7x} + \frac{2}{x^2}}$$

③

$$\lim_{x\to\infty} \frac{\overset{\nearrow \frac{3}{7}}{\frac{3}{7}} - \overset{\nearrow 0}{\frac{2}{7x}} + \overset{\nearrow 0}{\frac{1}{7x^2}}}{\underset{\searrow 1}{1} + \underset{\searrow 0}{\frac{1}{7x}} + \underset{\searrow 0}{\frac{2}{x^2}}} = \frac{\frac{3}{7}+0+0}{1+0+0} = \frac{3}{7}$$

Example

Calculate $\displaystyle\lim_{x\to\infty} \frac{1+\cos(x)}{10x}$.

Solution

$$\underset{+1}{-1} \le \underset{+1}{\cos(x)} \le \underset{+1}{1}$$

$$0 \le 1+\cos(x) \le 2.$$

$$\underset{\substack{\uparrow\\ f(x)}}{\frac{0}{10x}} \le \underset{\substack{\uparrow\\ g(x)}}{\frac{1+\cos(x)}{10x}} \le \underset{\substack{\uparrow\\ h(x)}}{\frac{2}{10x}}$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{0}{10x} = 0$$

$$\lim_{x \to \infty} h(x) = \lim_{x \to \infty} \frac{2}{10x} = 0$$

So by the Squeeze Lemma:

$$\lim_{x \to \infty} \frac{1 + \cos(x)}{10x} = 0.$$