1. Difference quotients.

2. When limits do & do not exist.

3. Calculating limits using algebra.

0. Limits

$$f(x) = \frac{\sin(x)}{x}.$$

- Limits allow us to determine what the function would do at a point if it were defined there.

e.g. $f(x) = \dfrac{\sin(x)}{x}$

is not defined at $x = 0$.

However:

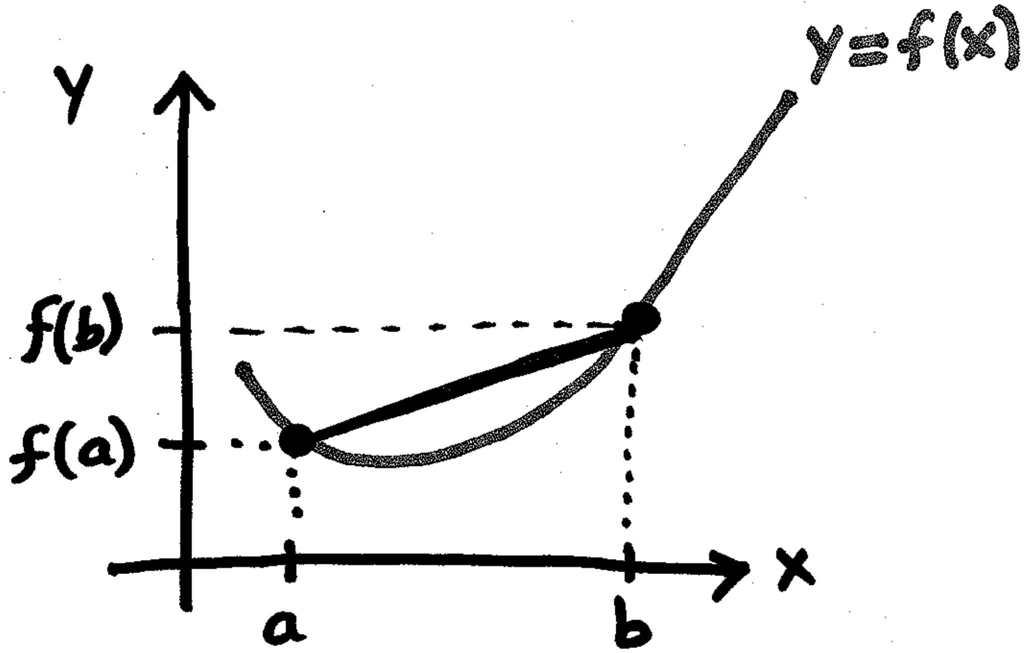
$$\lim_{x \to 0} f(x) = 1.$$

when x gets really close to $x = 0$

y-values on $y = f(x)$ get really close to $y = 1$

1. Difference Quotients

If $y = f(x)$ is a function and $(a, f(a))$ and $(b, f(b))$ are two points on the graph,

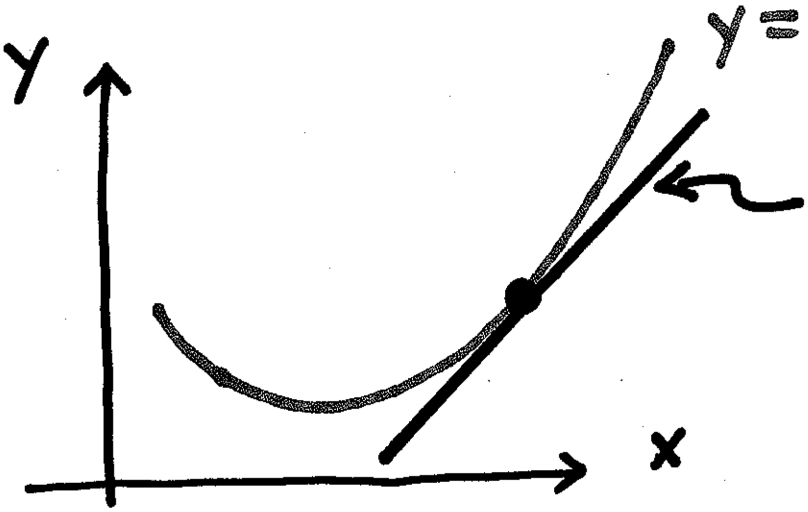


the slope of the line joining these points is:

$$\text{slope} = \frac{f(b) - f(a)}{b - a}.$$

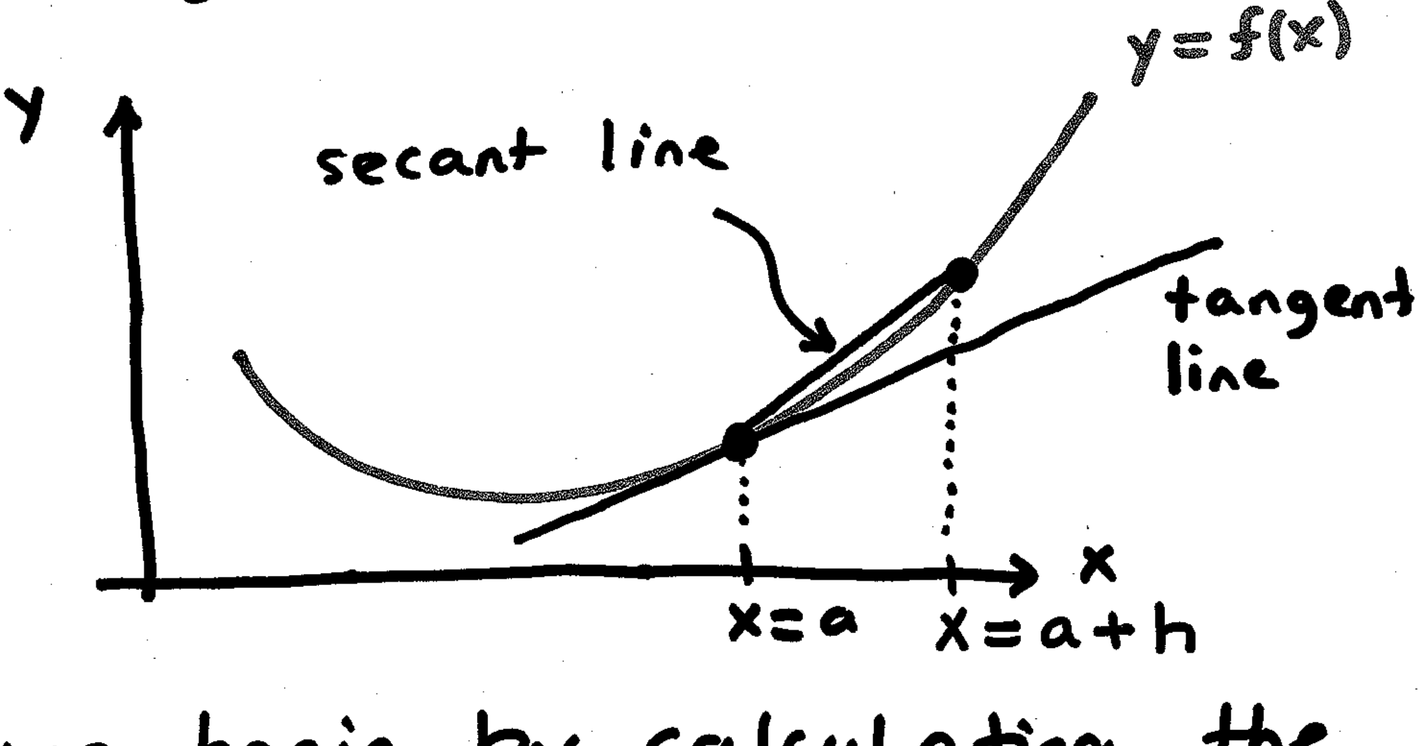
- Any quotient $\dfrac{\Delta \text{ function}}{\Delta x}$ is called a difference quotient.

Tangent Lines



tangent line meets the curve at one point in a tender caress.

To get the slope of the tangent line at $x = a$



we begin by calculating the slope of the secant line that joins $(a, f(a))$ and $(a+h, f(a+h))$ where $h > 0$.

$$\text{Slope of secant line} = \frac{f(a+h) - f(a)}{a+h - a}$$

As h gets smaller and smaller, the secant line starts to resemble the tangent line very closely. The slopes of the two lines get closer as h gets closer to zero.

$$\text{Slope of tangent line} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{a+h - a}$$

2. When Limits Do and Do Not Exist.

- First piece of notation:

$$\underbrace{x \to 4^{-}}_{\text{x gets closer and closer to } x=4.}$$

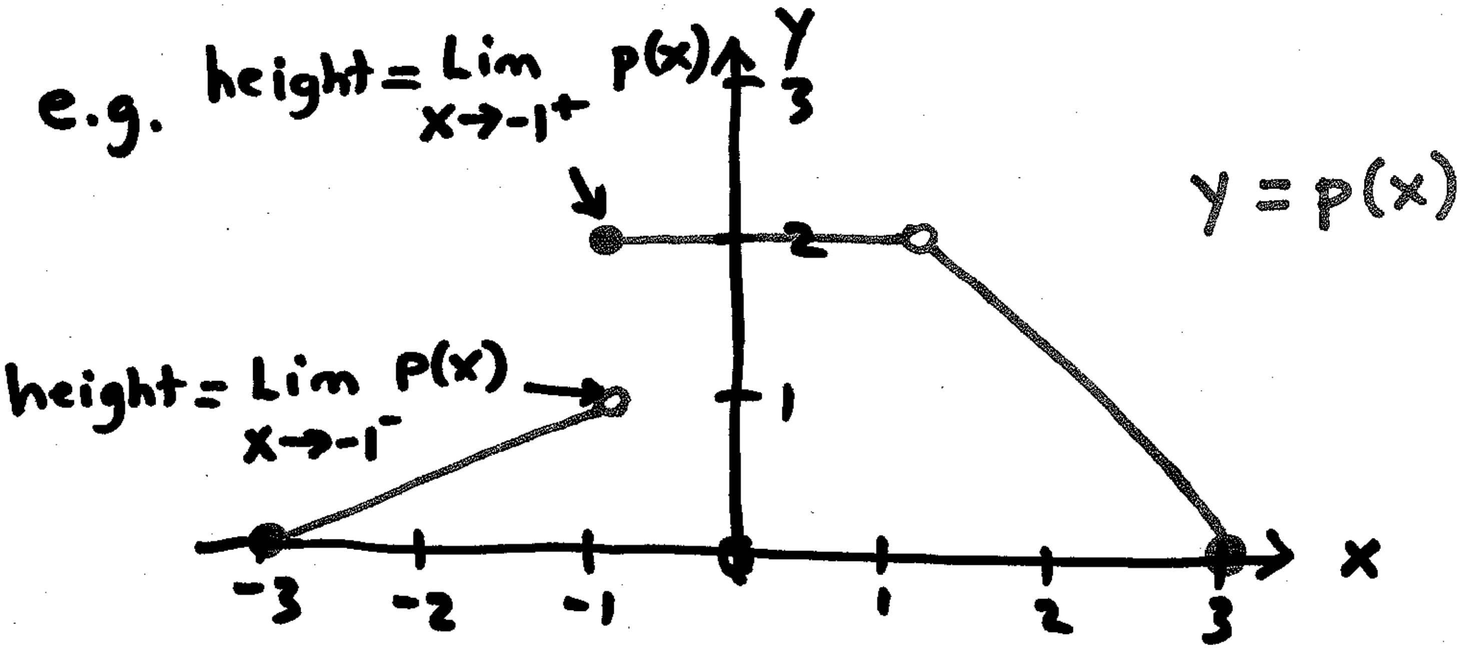
Always stay to the left of $x = 4$.

- Second piece of notation:

$$\underbrace{x \to 4^{+}}_{\text{x gets closer and closer to } x=4}$$

Always stay to the right of $x = 4$.

Left Hand and Right Hand Limits



<u>What is:</u>

$$\lim_{x \to -1^-} p(x) = 1$$

$$\lim_{x \to -1^+} p(x) = 2$$

$$\lim_{x \to 1^-} p(x) = 2$$

$$\lim_{x \to 1^{\cancel{+}-}} p(x) = 2$$

$$\lim_{x \to 1^+} p(x) = 2.$$

Definition: The limit of $f(x)$ as $x \to a$, which we write as: $\lim_{x \to a} f(x)$, exists if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$. That is, $\lim_{x \to a} f(x)$ exists if the right hand limit equals the left hand limit.

Back to previous answer:

$\lim_{x \to -1} p(x)$ does not exist.

$\lim_{x \to 1} p(x)$ exists and equals 2.

3. Calculating Limits Using Algebra.

Example.

$y = f(x) = x^2$. What is the slope of the tangent line at $a = 3$.

Solution

$$\text{Slope of secant line} = \frac{f(a+h) - f(a)}{a+h-a}$$

$$= \frac{(3+h)^2 - 3^2}{3+h-3}$$

$$= \frac{9 + 6h + h^2 - 9}{3+h-3}$$

$$= \frac{6h + h^2}{h} = \frac{h \cdot (6+h)}{h}.$$

So long as $h \neq 0$,

$$\text{Slope of secant line} = \frac{\cancel{h}(6+h)}{\cancel{h}} = 6+h.$$

As $h \to 0$, $6+h \to 6$.

$$\text{Slope of tangent line} = 6.$$