Outline

1. Integration by parts.
2. Integration by parts for definite integrals.

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① Do-over: Tonight 1212 DH
8-9pm or 9-10pm

② Final: Friday, May 8 1pm-4pm
Baker Hall A51

③ Gateway deadline: Friday 5pm.

1. Integration by Parts

$$\boxed{\int u \cdot v' \, dx = u \cdot v - \int u' \cdot v \cdot dx}$$

Where this comes from:

Product Rule: $\frac{d}{dx}(u \cdot v) = u' \cdot v + u \cdot v'$

Integrate: $\int \frac{d}{dx}(u \cdot v) dx = \int u' \cdot v \, dx + \int u \cdot v' dx$

$$u \cdot v = \int u' \cdot v \, dx + \int u \cdot v' dx$$

$$\boxed{u \cdot v - \int u' \cdot v \, dx = \int u \cdot v' \, dx}$$

Example

Evaluate: $\int x e^{-x} \, dx$

Solution:

$u = x \qquad u' = 1$ u' is simpler than u.

$v' = e^{-x} \qquad v = -e^{-x}$ v is no more complicated than v'.

$$\int u \cdot v' \, dx = u \cdot v - \int u' \cdot v \, dx$$

$$\int x \cdot e^{-x} \, dx = -x e^{-x} - \int (1)(-e^{-x}) \, dx$$

$$= -x e^{-x} + \underbrace{\int e^{-x} \, dx}$$

$$= -x e^{-x} + \; - e^{-x} + C$$

$$\int x \cdot e^{-x} \, dx = -x e^{-x} - e^{-x} + C.$$

Example

Evaluate $\int x^2 \cdot e^{x} \cdot dx.$

Solution

$u = e^x \qquad u' = e^x$

$v' = x^2 \qquad v = \frac{1}{3}x^3$

$$\int u \cdot v' \, dx = u \cdot v - \int u' \cdot v \, dx$$

$$\int x^2 e^x \, dx = \frac{1}{3}x^3 e^x - \underbrace{\int \frac{1}{3}x^3 e^x \, dx}$$

this is more complicated than the original integral, so we will try another choice of u and v'.

Try again:

$u = x^2 \qquad u' = 2x$

$v' = e^x \qquad v = e^x$

$$\int u v' \, dx = u \cdot v - \int u' v \, dx$$

$$\int x^2 e^x \, dx = x^2 e^x - \underbrace{\int 2x \cdot e^x \, dx}$$

use integration by parts a second time to integrate this.

$u = 2x$ $\qquad$ $u' = 2$

$v' = e^x$ $\qquad$ $v = e^x$

$$\int u v' \, dx = u \cdot v - \int u' \cdot v \, dx$$

$$\int 2x \cdot e^x \cdot dx = 2x \cdot e^x - \int 2 \cdot e^x \, dx$$

$$= 2xe^x - 2e^x + C$$

Now substitute this back in to what we were calculating.

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$$

$$= x^2 e^x - \left(2x e^x - 2e^x\right) + C$$

$$= x^2 e^x - 2x e^x + 2e^x + C$$

2. Integration by Parts with Definite Integrals

- Procedure:

① Write out the integral as an indefinite integral (no limits).

② Use integration by parts on the indefinite integral to find the antiderivative formula.

③ Plug the limits of integration into the antiderivative and subtract.

Example

Evaluate $\int_1^2 \ln(x)\,dx$.

Solution

① $\int \ln(x)\,dx = \int \ln(x) \cdot 1\,dx$

$u = \ln(x) \qquad u' = 1/x$

$v' = 1 \qquad v = x$

② $\int u v'\,dx = uv - \int u' v\,dx$

$\int 1 \cdot \ln(x)\,dx = x \cdot \ln(x) - \int \frac{1}{x} x\,dx$

$= x \cdot \ln(x) - \int 1\,dx$

$\int \ln(x)\,dx = x \cdot \ln(x) - x + C$

③ $\int_1^2 \ln(x)\,dx =$ [redacted]

$$\int_1^2 \ln(x)\,dx = 2\cdot\ln(2) - 2 + C$$

$$- \left(1\cdot\ln(1) - 1 + C \right)$$

$$= 2\cdot\ln(2) - 1.$$

$$\approx 0.38$$

<u>Example</u>

Calculate: $\int_1^2 \sqrt{y}\cdot\ln(y)\cdot dy.$

<u>Solution</u>

① $\int \sqrt{y}\cdot\ln(y)\,dy$

$u = \ln(y)$ $\qquad\qquad u' = 1/y$

$v' = \sqrt{y} = y^{1/2}$ $\qquad v = \frac{2}{3}y^{3/2}$

$$\int u \cdot v' \, dx = u \cdot v - \int u' \cdot v \, dx$$

$$\int \sqrt{y} \cdot \ln(y) \, dy = \frac{2}{3} y^{3/2} \cdot \ln(y)$$

$$- \int \frac{2}{3} y^{3/2} \cdot \frac{1}{y} \, dy$$

$$= \frac{2}{3} y^{3/2} \cdot \ln(y) - \int \frac{2}{3} y^{1/2} \, dy$$

$$= \frac{2}{3} y^{3/2} \cdot \ln(y) - \frac{2}{3} \cdot \frac{2}{3} \cdot y^{3/2} + C$$

$$= \frac{2}{3} y^{3/2} \cdot \ln(y) - \frac{4}{9} y^{3/2} + C$$

③ $$\int_1^2 \sqrt{y} \cdot \ln(y) \, dy = \frac{2}{3} (2)^{3/2} \ln(2)$$

$$- \frac{4}{9} (2)^{3/2} - \Big(\frac{2}{3} (1)^{3/2} \ln(1)$$

$$- \frac{4}{9} (1)^{3/2} \Big)$$

$$\approx 0.49437$$