Outline

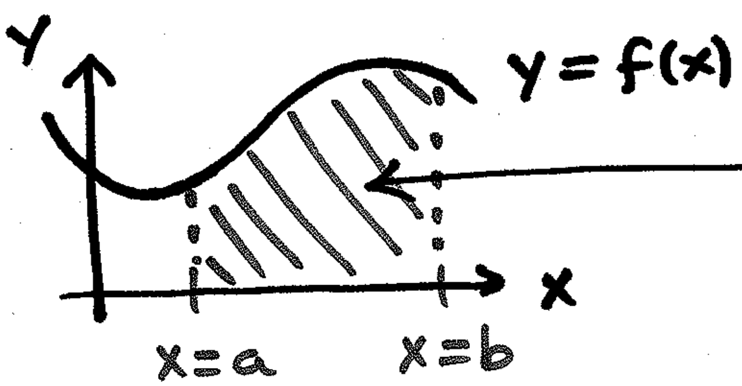
1. Properties of definite integrals.
2. Evaluating definite integrals.
3. Functions defined by integrals.
4. Second Fundamental Theorem.

1. Properties of Definite Integrals

- Limit of the Riemann sum:

$$\sum_{k=0}^{N-1} f(a + k \cdot \Delta x) \cdot \Delta x$$

as either $N \to \infty$ or $\Delta x \to 0$, where $\Delta x = \dfrac{b-a}{N}$.



Exact area $= \int_a^b f(x)\,dx$

- An indefinite integral is written: $\int f(x)\,dx$ (no limits of integration) and means the most general antiderivative of $f(x)$. (i.e. includes $+C$).

- **Definite integral:** Produces a number as the result.

- **Indefinite integral:** Produces a formula as the result.

2. Evaluating Definite Integrals

1st Fundamental Theorem: If $F(x)$ is antiderivative of $f(x)$ then:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

- To find the area under a curve:

  ① Find an antiderivative for $f(x)$.

  ② Plug $x=b$ and $x=a$ into the antiderivative, $F(x)$.

  ③ Area $= \int_a^b f(x)\,dx = F(b) - F(a)$.

Example

Find the area between $f(x) = x^3 - x$ and the x-axis for:

(i) $a = -1$ and $b = 0$

(ii) $a = 0$ and $b = 1$

(iii) $a = -1$ and $b = 1$.

Solution

(i) $\int_{-1}^{0} (x^3 - x)\,dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^{0}$

$= \frac{1}{4}(0)^4 - \frac{1}{2}(0)^2 - \left(\frac{1}{4}(-1)^4 - \frac{1}{2}(-1)^2\right)$

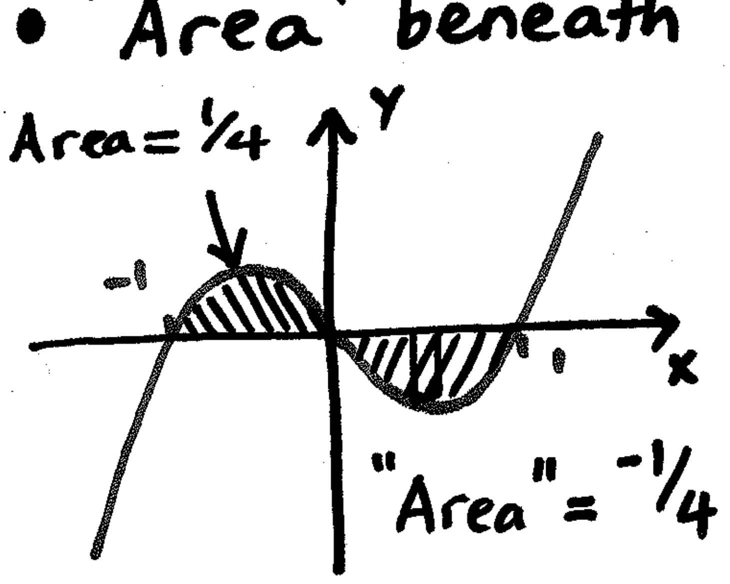
$= \frac{1}{4}.$

(ii) $\int_{0}^{1} (x^3 - x)\,dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2 + C\right]_{0}^{1}$

$= \frac{1}{4}(1)^4 - \frac{1}{2}(1)^2 + C - \left(\frac{1}{4}(0)^4 - \frac{1}{2}(0)^2 + C\right)$

$= -\frac{1}{4}.$ ← no trace of $+C$ because the two $+C$'s cancel.

- "Area" beneath the x-axis is regarded as negative.



- For our clan, "area" will always refer to the positive quantity,

$$\text{Area} = \left| \int_a^b f(x)\,dx \right|$$

- The "value of the definite integral" can be positive or negative.

(iii) Can use the additive property of definite integrals:

$$\int_a^b f(x)\,dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

to get:

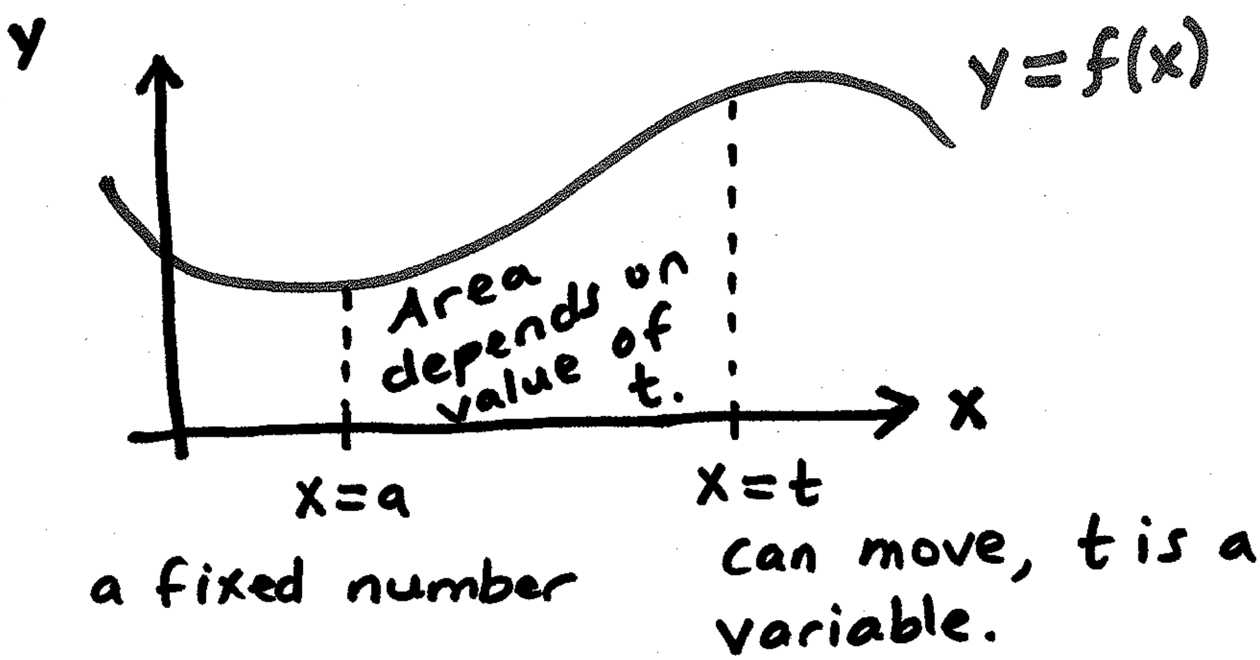
$$\int_{-1}^{1} (x^3-x)dx = \int_{-1}^{0} (x^3-x)dx + \int_0^1 (x^3-x)\,dx$$

$$= \quad 1/4 \quad + \quad -1/4$$

$$= \; 0.$$

3. Functions Defined by Integrals.

- Need to keep track of x and t very carefully.
- Idea: Make one of the limits of integration a variable "t."



- Define a new function $g(t)$ which gives the area between $x=a$ and $x=t$.

$$g(t) = \int_a^{t} f(x)\, dx$$

the variable we can work with.

not a variable we can access or manipulate.

4. Second Fundamental Theorem of Calculus.

Basic Version: Tells you how to take the derivative of a function defined by an integral.

$$g(t) = \int_a^t f(x)\,dx$$

then:

$$g'(t) = f(t)$$

(Take function being integrated and replace x's by t's.)

Advanced version:

$$g(t) = \int_{p(t)}^{q(t)} f(x)\,dx.$$

Then:

$$g'(t) = f\big(q(t)\big)\cdot q'(t) - f\big(p(t)\big)\cdot p'(t)$$

NOTE: $\frac{d}{dt}\left(\int_{a}^{b} f(x)\,dx\right) = 0$

constant (b)

constant (a)