Outline

1. Differential equations.
2. L'Hôpital's rule.
3. Manipulating limits for L'Hôpital's rule.

—||—

Test #2: Friday.

1. Differential Equations

function we want.

Differential Equation : $$\frac{dy}{dt} = K \cdot y(t)$$

constant

constant

Solution : $$y(t) = A \cdot e^{kt}$$

constant

function

Differential Equation : $$\frac{dy}{dt} = K \cdot \left[y(t) - B \right]$$

constant

another constant.

Solution : $$y(t) = A \cdot e^{kt} + B$$

Why does this work?

Start out with:

$$y(t) = A \cdot e^{kt} + B.$$

On one hand:

$$\frac{dy}{dt} = A \cdot e^{kt} \cdot k + 0$$

$$= k \cdot A \cdot e^{kt} \quad \ldots (*)$$

On the other hand:

$$k \cdot [y(t) - B] = k \cdot [Ae^{kt} + B - B]$$

$$= k \cdot A \cdot e^{kt} \quad \ldots (**)$$

Get:

$$\frac{dy}{dt} = k \cdot [y(t) - B]$$

whenever $y(t) = A \cdot e^{kt} + B.$

Example

A dead body is discovered at 8 am and its temperature is 72°F. One hour later the temperature is 68°F. The body was found in a refrigerator that maintained a steady temperature of 30°F. When did the person die?

Solution

$H(t)$ = temperature of body t hours ~~after death~~. after 8am in the morning.

Newton's Law of Cooling: $$\frac{dH}{dt} = K \cdot \left[H(t) - 30 \right]$$

We want: Value of t for which

$$H(t) = 98.6.$$

Plan: ① Find a formula for $H(t)$.

② Set $H(t) = 98.6$

③ Solve for t.

Step ①: $H(t) = A \cdot e^{kt} + 30$

Have: $H(1) = 68$ $H(0) = 72$.

Use these to figure out A & k.

<u>$H(0) = 72$</u> $72 = A \cdot e^{k(0)} + 30$

$$\boxed{42 = A}$$

<u>$H(1) = 68$</u> $68 = 42 \cdot e^{k(1)} + 30$

$$38 = 42 e^{k}$$

$$\frac{38}{42} = e^{k}$$

$$k = \ln\left(\frac{38}{42}\right) \approx -0.1008$$

Formula for $H(t)$:

$$H(t) = 42e^{-0.1008t} + 30$$

Step ②: $$98.6 = 42e^{-0.1008t} + 30$$

Step ③: $$68.6 = 42e^{-0.1008t}$$

$$\frac{68.6}{42} = e^{-0.1008t}$$

$$-0.1008t = \ln\left(\frac{68.6}{42}\right)$$

$$t = \frac{1}{-0.1008} \ln\left(\frac{68.6}{42}\right)$$

$$\approx -4.86$$

This would be about 3:10 am.

2. L'Hôpital's Rule

- Simplify limits using derivatives.

Theorem (v.1): If $\lim_{x \to a} f(x) = 0$

and $\lim_{x \to a} g(x) = 0$

then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \underbrace{\lim_{x \to a} \frac{f'(x)}{g'(x)}}_{\text{often much easier to calculate than original limit.}}$$

Example

Calculate: $\lim_{\theta \to 0} \frac{\theta + \tan(\theta)}{\sin(\theta)}$.

Solution

$$\left.\begin{aligned}\lim_{\theta\to 0} \theta + \tan(\theta) &= 0 \\ \lim_{\theta\to 0} \sin(\theta) &= 0\end{aligned}\right\}$$ So L'Hôpital's rule applies.

$$\lim_{\theta\to 0} \frac{\theta + \tan(\theta)}{\sin(\theta)} = \lim_{\theta\to 0} \frac{1+\sec^2(\theta)}{\cos(\theta)}$$

$$= \frac{1+1}{1} = 2$$

Example

Let n, m be positive constants.

Calculate: $\displaystyle\lim_{t\to 0} \frac{\cos(mt) - \cos(nt)}{t^2}$.

Solution

$$\left.\begin{aligned}\lim_{t\to 0} \cos(mt) - \cos(nt) &= 0 \\ \lim_{t\to 0} t^2 &= 0\end{aligned}\right\}$$ So L'Hôpital's rule applies.

$$\lim_{t \to 0} \frac{\cos(mt) - \cos(nt)}{t^2}$$

$$= \lim_{t \to 0} \frac{-\sin(mt) \cdot m + \sin(nt) \cdot n}{2t}$$

$$= \lim_{t \to 0} \frac{-\cos(mt) . m^2 + n^2 . \cos(nt)}{2}$$

$$= \frac{-m^2 + n^2}{2}$$