Outline

1. New derivative formulas.
2. Differential equations.
3. Solutions of differential equations.

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Next test: Friday 3/20/09.

1. Derivative Formulas

Inverse Trigonometry Functions

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$$

Example

$$f(x) = \tan^{-1}\left(\ln(x) + \sin(x)\right)$$

then:

$$f'(x) = \frac{1}{1+(\ln(x)+\sin(x))^2}\cdot\left(\frac{1}{x}+\cos(x)\right)$$

- **Hyperbolic Functions**

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

"hyperbolic sin"

"shine"

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

"hyperbolic cos"

" cosh "

$$\frac{d}{dx}(\sinh(x)) = \cosh(x)$$

$$\frac{d}{dx}(\cosh(x)) = \sinh(x).$$

Example

$$f(x) = \sinh(3^x + 2)$$

$$f'(x) = \cosh(3^x + 2) \cdot (\ln(3) \cdot 3^x + 0)$$

2. Differential Equations

- Idea is to specify a function by giving:
  - ☐ An equation for the derivative of the function
  - ☐ One value of the function.
- Goal is usually to find the formula for the function.

Example (Chemistry)

- Law of Mass action:

  Rate of a chemical reaction is proportional to the concentration of the reagent.

- In math terms:

$C(t)$ = concentration of the reagent after t minutes

$$\underset{\substack{\uparrow \\ \text{rate of} \\ \text{reaction}}}{C'(t)} = \underset{\substack{\uparrow \\ \text{constant} \\ \text{of proportionality}}}{K} \cdot \underset{\substack{\nwarrow \\ \text{concentration}}}{C(t)}$$

- Goal: Find an explicit formula for $C(t)$.

3. Finding Solutions for Differential Equations

- A solution is a formula for the function whose derivative is given in the differential equation.

- Goal: Find this solution.

- To find the solution of:

$$C'(t) = k \cdot C(t)$$

note that:

$$\text{If } f(t) = A \cdot e^{kt}$$

constant numbers

$e \approx 2.718$

$$\text{then: } f'(t) = A \cdot k \cdot e^{kt}$$

$$\text{or: } f'(t) = k \cdot \left(A \cdot e^{kt} \right)$$

$$\text{or: } f'(t) = k \cdot f(t).$$

- Summary:

  If we have a differential equation that looks like:

  $$f'(t) = k \cdot f(t)$$

  then the solution of this is:

  $$f(t) = A \cdot e^{k \cdot t}$$

Example (Biology).

In the absence of constraints (e.g. limited food, space, etc.), a population normally grows at a rate proportional to the size of the population.

A herd of deer enjoys population growth at a continuous rate of 3% per year. At $t=0$ there are 100 deer. Find the number of deer 3 years later.

Solution

k = continous growth rate (expressed as a decimal).

$= 0.03$.

Let $P(t)$ be the number of deer after t years.

$$\underset{\substack{\uparrow \\ \text{growth rate}}}{P'(t)} = \underset{\substack{\uparrow \\ k}}{0.03} \cdot \underset{\substack{\uparrow \\ \text{\# of deer.}}}{P(t)}$$

Formula for $P(t)$:

$$P(t) = A \cdot e^{0.03t}$$

To find A, plug in $t=0$ and $P(t) = 100$.

$$100 = A \cdot e^{(0.03)(0)}$$

$$100 = A$$

So, complete formula for $P(t)$ is:

$$P(t) = 100 \cdot e^{0.03 \cdot t}$$

When $t = 3$:

$$P(3) = 100 \cdot e^{(0.03)(3)} = 109.42$$

- **Note:** The relative growth rate is:

$$\frac{P'(t)}{P(t)} = \frac{\text{derivative}}{\text{function}}$$

Solving a More Complicated Differential Equation

$R(t)$ = # of words remembered after t days.

$$\frac{dR}{dt} = \frac{1}{2}R(t) - 3.5$$

$$R'(t) = \frac{1}{2}\left(R(t) \boxed{-7}\right)$$

only thing different from before.

Solution: $R(t) = \underset{\substack{\uparrow \\ \text{constant}}}{A} \cdot e^{\frac{1}{2}t} - (-7)$

$$R(t) = A \cdot e^{\frac{1}{2}t} + 7$$

$R(0) = 10$ To determine A:

$$10 = A \cdot e^{\frac{1}{2}(0)} + 7$$

$$10 = A + 7$$

$$3 = A$$

Complete solution:

$$R(t) = 3e^{\frac{1}{2}t} + 7$$