Outline

1. Solving equations with exponentials and logarithms.

2. Derivatives of exponentials and logarithms.

3. Differential equations.

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HW due Tuesday

No class Thursday or Friday.

1. Solving Equations

Properties:

$$\log(a \cdot b) = \log(a) + \log(b)$$

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$$

$$\log(a^x) = x \cdot \log(a)$$

$$e^{\ln(x)} = x \qquad 10^{\log(x)} = x$$

Example

Solve: $\log(x) + \log(x+21) = 2.$

Solution

$$\log(x \cdot (x+21)) = 2$$

$$10^{\log(x \cdot (x+21))} = 10^2$$

$$x \cdot (x+21) = 100$$

$$x^2 + 21x - 100 = 0$$

$$x = \frac{-21 \pm \sqrt{21^2 - 4(1)(-100)}}{(2)(1)}$$

$$= -25, 4$$

<u>Note:</u> The domain of $f(x) = \log(x)$ and of $g(x) = \ln(x)$ consists of:

$$\boxed{x > 0.}$$

This means that out of $x = -25$ and $x = 4$, only $x = 4$ will be a valid solution of the equation:

$$\log(x) + \log(x+21) = 2.$$

Handout 16: Manipulating Equations Using Logarithms

Solve each of the following equations for x. You should use algebra and logarithms (where appropriate) to solve the equations. Show all of your work.

You should not use your calculator to solve these equations except for performing arithmetic and working out the numerical values of logarithms. You can use either common or natural logarithms.

(a) $3 \cdot 10^x = 900$

$$10^x = \frac{900}{3}$$

$$\log(10^x) = \log(300)$$

$$x = \log(300)$$

$$x \approx 2.477$$

(b) $10^{\ln(x)} = 100$

$$\log(10^{\ln(x)}) = \log(100)$$

$$\ln(x) = 2$$

$$e^{\ln(x)} = e^2$$

$$x = e^2$$

$$\approx 7.38$$

(c) $7 \cdot e^{3x} = 2000$

$$e^{3x} = \frac{2000}{7}$$

$$\ln(e^{3x}) = \ln(2000/7)$$

$$3x = \ln(2000/7)$$

$$x = \frac{1}{3}\ln(2000/7)$$

$$x \approx 1.885$$

(d) $e^{\ln(4x+5)} = 3$

$$4x + 5 = 3$$
$$4x = -2$$
$$x = -1/2$$

(f) $5^{x+1} = 22e^{3x}$

$$\ln(5^{x+1}) = \ln(22) + \ln(e^{3x})$$
$$(x+1)\cdot\ln(5) = \ln(22) + 3x$$
$$x+1 = \frac{\ln(22)}{\ln(5)} + \frac{3}{\ln(5)}x$$
$$x - \frac{3}{\ln(5)}x = \frac{\ln(22)}{\ln(5)} - 1$$
$$x\cdot\left(1 - 3/\ln(5)\right) = \frac{\ln(22)}{\ln(5)} - 1$$

$$x = \frac{\frac{\ln(22)}{\ln(5)} - 1}{1 - 3/\ln(5)}$$
$$\approx -1.065$$

(g) $(1+x)^5 = 777$

$$\left((1+x)^5\right)^{1/5} = 777^{1/5}$$
$$1+x = 777^{1/5}$$
$$x = 777^{1/5} - 1$$
$$\approx 2.75$$

2. Derivatives of Logs and Exponentials

(a) Exponential Functions

$f(x) = A \cdot B^x$ then provided $B > 0$:

$$\boxed{f'(x) = A \cdot B^x \cdot \ln(B).}$$

Special case: $f(x) = 1 \cdot e^x = e^x$

$$f'(x) = 1 \cdot e^x \cdot \ln(e)$$
$$= e^x$$

Example

Find a formula for $p'(t)$ where:

$$p(t) = \frac{1}{1 + a \cdot e^{-kt}}$$

where a and k are positive constants.

$$a \cdot e^{(-kt)}$$

Use chain Rule

Solution

$$p'(t) = \frac{(0)\cdot\left(1+a\cdot e^{-kt}\right) - (1)\cdot\left(0 + a\cdot e^{(-kt)}\cdot(-k)\right)}{\left(1+a\cdot e^{-kt}\right)^2}$$

$$= \frac{k\cdot a\cdot e^{-kt}}{\left(1+a\cdot e^{-kt}\right)^2}$$

(b) Logarithms

$$f(x) = \log(x) \qquad\qquad g(x) = \ln(x)$$

$$f(x) = \frac{\ln(x)}{\ln(10)} = \underbrace{\frac{1}{\ln(10)}}_{\text{constant}}\cdot\ln(x)$$

$$f'(x) = \frac{1}{\ln(10)}\cdot\frac{1}{x} \qquad\qquad g'(x) = \frac{1}{x}$$