Outline

1. Derivative as a function.
2. Units and interpretation of derivative.
3. Graph of derivative.

1. <u>Derivative is a Function.</u>

- If $f(x)$ is the function (input variable, x) then we can also find a function/formula for the <u>derivative</u> written as $f'(x)$ (also with input variable, x).
- Formula for $f'(x)$ is:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- h is the variable that changes, and x is just along for the ride.

Example

Calculate $f'(x)$ given:

$$f(x) = x + \frac{1}{x}.$$

Solution

① Set up the difference quotient.

$$\frac{f(x+h) - f(x)}{h} = \frac{x+h + \frac{1}{x+h} - \left(x + \frac{1}{x}\right)}{h}$$

② Simplify the numerator of the difference quotient.

<u>Goal</u>: Cancel everything in the numerator that doesn't have a factor of h.

$$\frac{x+h+\frac{1}{x+h}-x-\frac{1}{x}}{h}$$

$$=\frac{h+\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$=\frac{h+\frac{x-(x+h)}{x\cdot(x+h)}}{h}$$

$$=\frac{h+\frac{x-x-h}{x(x+h)}}{h}$$

$$=\frac{h\cdot 1+h\cdot\frac{-1}{x(x+h)}}{h}$$

③ Cancel h's in numerator with h in denominator (assuming $h \neq 0$).

$$= 1 + \frac{-1}{x \cdot (x+h)}$$

④ When h in denominator is gone, take limit as $h \to 0$.

$$f'(x) = \lim_{h \to 0} 1 + \frac{-1}{x \cdot (x+h)}$$

$$= 1 + \frac{-1}{x^2}$$

Example

Calculate $f'(x)$ when:

$$f(x) = \sqrt{x}.$$

Solution

① $$\frac{f(x+h)-f(x)}{h} = \frac{\overset{a}{\sqrt{x+h}} - \overset{b}{\sqrt{x}}}{h}$$

② Difference of two squares:

$$a^2 - b^2 = (a-b)\underbrace{(a+b)}$$

multiply top & bottom by $a+b$ to get rid of square roots in numerator.

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h\cdot(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h\cdot(\sqrt{x+h}+\sqrt{x})}$$

$$= \frac{x + h - x}{h \cdot (\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{h \cdot 1}{h \cdot (\sqrt{x+h} + \sqrt{x})}$$

③ $$= \frac{1}{\sqrt{x+h} + \sqrt{x}}, \quad h \neq 0.$$

④ $$f'(x) = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

2. <u>The Units and Interpretation of the Derivative</u>

Example

It costs $f(x)$ dollars to extract x ounces of weapons grade uranium from a ton of uranium ore (obtained from Niger).

(a) what are the units of:

(i) x

(ii) $f(x)$

(iii) $f'(x)$.

(b) What is the practical meaning of:

$$f'(8) = 17\,000 \text{ ?}$$

Solution

(a) (i) Units of input x are ounces.

(ii) Units of output $f(x)$ are dollars.

(iii) Units of derivative $= \dfrac{\text{units of output}}{\text{units of input}}$

$= \dfrac{\text{dollars}}{\text{ounces}}$

or dollars per ounce.

(b) $f'(8) = 17\,000$

If uranium extraction is increased by 1 from $\boxed{\text{8 ounces to 9}}$ ounces then the cost is increased by approximately $\boxed{\$17{,}000.}$