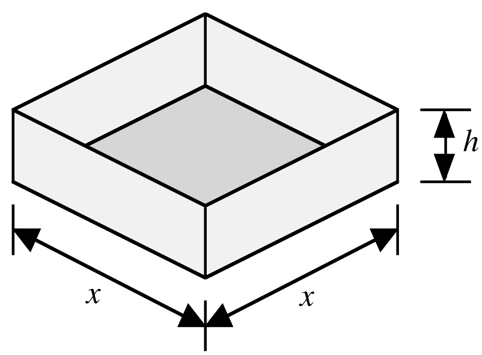
Solutions to Homework #9

Problems from Pages 232-236 (Section 4.5)

10. In this problem, we will make the assumption that the amount of material used to construct the box is directly related to the surface area of the box. Our objective will be to find the dimensions of the box that minimize the surface area.



The surface area of the box, S, is given by the formula: $S = 4xh + x^2$.

To eliminate h from the formula for surface area, we can use the fact that the volume of the box is supposed to be 32,000 cubic centimeters.

$$\text{Volume} = x^2 \cdot h = 32000$$

$$h = \frac{32000}{x^2}$$

Substituting this into the formula for the surface area and simplifying gives:

$$S = \frac{128000}{x} + x^2.$$

To find the critical point(s) of surface area we will take the derivative of S with respect to x, set the derivative equal to zero and solve for x. Doing this:

$$\frac{dS}{dx} = \frac{-128000}{x^2} + 2x = 0$$

$$x^3 = 64000$$

$$x = 40 \text{ centimeters.}$$

To ensure that this is a local minimum for surface area, we can evaluate the second derivative with $x = 40$. Doing this gives:

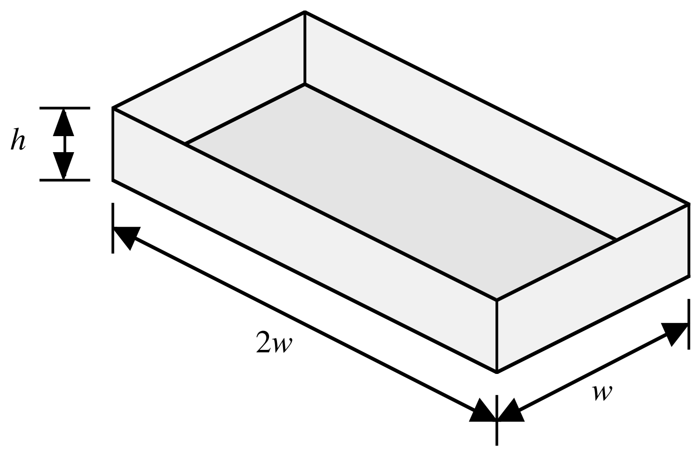
$$\left.\frac{d^2S}{dx^2}\right|_{x=4000} = \frac{256000}{(40)^3} + 2 > 0.$$

As the second derivative is positive, the critical point is a local minimum. To conclude this problem, we will calculate $h = 32000/(40)^2 = 20$ centimeters.

The dimensions of the box that will minimize the material used are 40cm for each side of the base, and a height of 20cm.

12. In this problem we will start by setting up a function for the cost, C, of the box (in dollars). This cost is the function that we will try to minimize.

The diagram shown below is a diagram of the box. Note that as the length of the box is twice the width, the length shown in this diagram is $2w$, w being the width of the box.



The material for the base costs \$10 per square meter and the material for the sides costs \$6 per square meter. Incorporating this information with the diagram shown above gives the following formula for the cost of the box:

$$C = 20w^2 + 24wh + 12wh = 20w^2 + 36wh.$$

To eliminate h from the cost function, we will exploit the fact that the volume of the box must be 10 cubic meters. This means that:

$$2w^2 \cdot h = 10$$

$$h = \frac{10}{2w^2} = \frac{5}{w^2}.$$

Substituting this into the cost function gives:

$$C = 20w^2 + \frac{180}{w}.$$

To find the critical point(s) we will take the derivative of C with respect to w, set this equation equal to zero and solve for w. Doing this:

$$\frac{dC}{dw} = 40w - \frac{180}{w^2} = 0.$$

Solving this equation for w gives $w = (180/40)^{1/3} \approx 1.65$ meters. To confirm that this value minimizes cost, we can substitute $w = (180/40)^{1/3}$ into the second derivative of cost:

$$\frac{d^2C}{dw^2} = 40 + \frac{360}{(1.65)^2} > 0.$$

As the second derivative is positive, a box with width of approximately 1.65 meters, a length of approximately 3.3 meters and a height of 1.83 meters will minimize cost.

28. In this problem, the function that we are trying to minimize is:

$$E(v) = av^3 \cdot \frac{L}{v-u}.$$

(a) We will calculate the first derivative of $E(v)$ with respect to v, set this equal to zero and then solve for v. Doing this:

$$E'(v) = \frac{3av^2L \cdot (v-u) - av^3L}{(v-u)^2} = aL\frac{2v^3 - 3uv^2}{(v-u)^2} = aL\frac{v^2 \cdot (2v-3u)}{(v-u)^2} = 0.$$

The solutions of this equation are $v = 0$ and $v = 1.5u$. Note that the first derivative is undefined when $u = v$ so that we have three critical points.

Note that in the problem we are told to assume that $0 < u < v$, so only one of the critical points ($v = 1.5u$) needs to be considered. Applying the first derivative test (and noting that $2v - 3u$ is negative to the left of $v = 1.5u$ but positive to the right of $v = 1.5u$) gives that the critical point $v = 1.5u$ is a local minimum.

(b) To sketch a plausible graph of $y = E(v)$, we need to note where the important features of the graph are located.

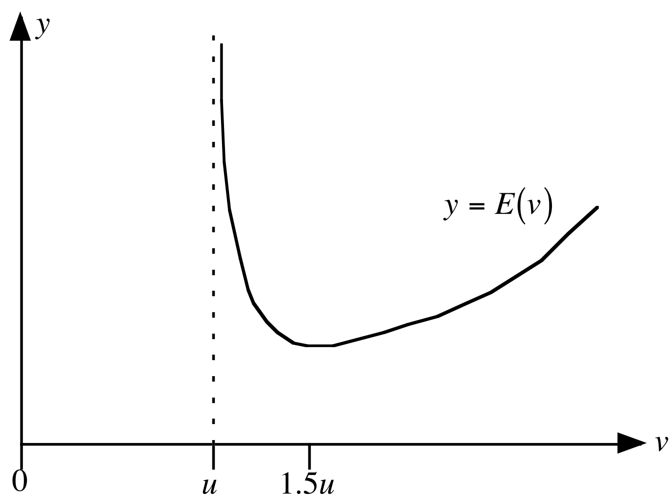
Domain: As we are told that $v > u > 0$, we only need to consider values of v that are greater than some positive constant u.

Intercepts: The only intercepts are both located at the origin (0, 0).

Asymptotes: There is a vertical asymptote at $v = u$. There are no horizontal asymptotes.

Turning points: The only turning point is a local minimum at $v = 1.5u$.

Drawing a graph that is consistent with the information listed above will produce a graph that will look something like the picture shown below.



38. **(a)** We are told to assume that the demand function is linear. Terry found that when he sold necklaces for \$10, he sold 20 necklaces. For every \$1 he raises his price, two fewer necklaces are sold. If we use x to represent the price (in dollars) then the number of necklaces sold, $d(x)$, will be:

$$d(x) = -2x + 40 \text{ necklaces.}$$

(b) The revenue function is $R(x) = x \cdot d(x) = -2x^2 + 40x$. Each necklace costs \$6 to make so that cost function is $C(x) = 6 \cdot d(x) = -12x + 240$. As profit is equal to revenue minus cost, the profit function will be:

$$p(x) = -2x^2 + 40x - (-12x + 240) = -2x^2 + 52x - 240.$$

To find the price that will give maximum profit, we will take the derivative of the profit function and set the derivative equal to zero.

$$p'(x) = -4x + 52 = 0.$$

The solution is $x = 13$. The second derivative is negative, confirming that this is a local maximum. So, Terry should raise his price to \$13 to maximize his profits.

Problems from Pages 240-241 (Section 4.6)

2. (This solution will be added as soon as the graph from the textbook can be scanned and annotated.)

6. To find the solution of the equation $x^5 - 2 = 0$ starting with the guess $x_1 = -1$ we will write:

$$f(x) = x^5 + 2 \quad \text{and} \quad f'(x) = 5x^4,$$

and summarize the computations needed to implement Newton's method in the table shown below.

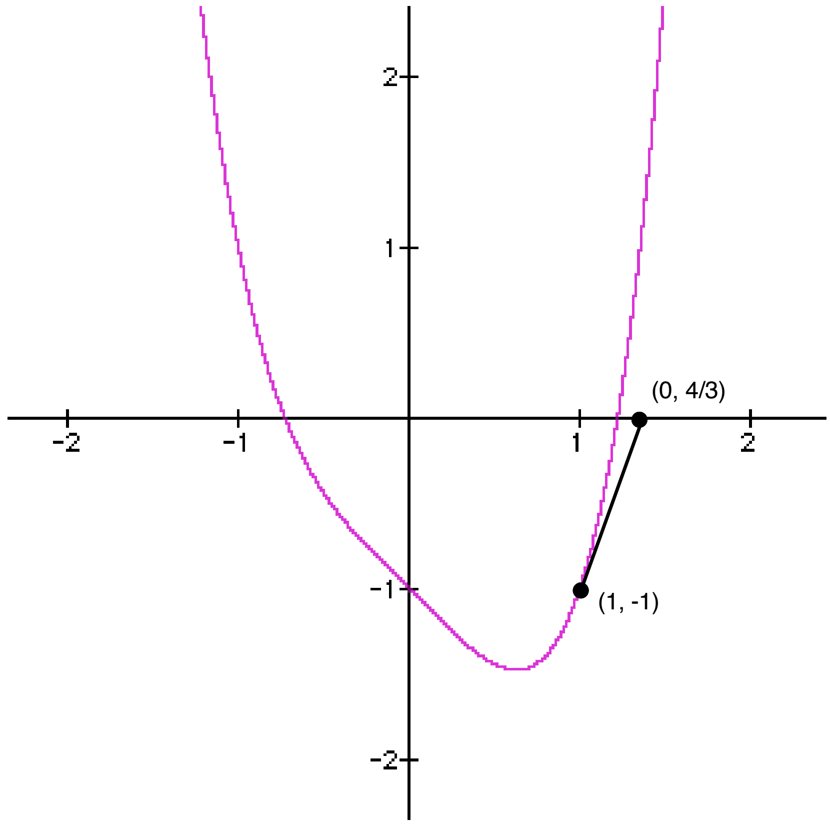
n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
1	−1	1	5	1.2
2	−1.2	−0.48832	10.368	−1.152901235
3	−1.152901235			

The third approximation of the solution to the equation is $x = -1.152901235$.

8. We will set $f(x) = x^4 - x - 1$ and use $f'(x) = 4x^3 - 1$ and $x_1 = -1$ so that:

$$x_2 = x_1 - f(x_1)/f'(x_1) = 4/3.$$

As the diagram given on the next page shows, the tangent line begins at the point (−1, 1) and then continues until it reaches the x-axis at the point (4/3, 0).



26. The local minimums of the function $f(x) = x^2 + \sin(x)$ will occur at points where the first derivative is equal to zero:

$$f'(x) = 2x + \cos(x) = 0.$$

This is the solution that we must solve using Newton's method.

To try to avoid confusion let's write $g(x) = 2x + \cos(x)$ and $g'(x) = 2 - \sin(x)$. To use Newton's method to solve the equation $g(x) = 0$, we will guess $x_1 = -0.5$ (you need to guess something – there are lots of other x_1 values that you could guess) and summarize the calculations involved in Newton's method in the following table.

n	x_n	$g(x_n)$	$g'(x_n)$	x_{n+1}
1	–0.5	–0.122417	2.479425	–0.450627
2	–0.450627	–0.001079	2.435530	–0.450184
3	–0.450184	–0.0000009	2.435131	–0.450184
4	–0.450184			

As the final two entries in the last column of the table agree to six decimal places (the level of accuracy that we are supposed to find), the x-value of the minimum value of $f(x)$ will occur at the point where $x = -0.450184$.

Problems from Pages 246-247 (Section 4.7)

6. To find the most general anti-derivative of the function:

$$f(x) = \sqrt[4]{x^3} + \sqrt[3]{x^4}$$

we will rewrite the function $f(x)$ in a form that is easier to work with:

$$f(x) = x^{\frac{3}{4}} + x^{\frac{4}{3}}.$$

Applying the power rule for anti-derivatives to each term and adding "+C" to the end gives the most general anti-derivative:

$$F(x) = \frac{x^{\frac{3}{4}+1}}{\frac{3}{4}+1} + \frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1} + C = \frac{4}{7}x^{\frac{7}{4}} + \frac{3}{7}x^{\frac{7}{3}} + C.$$

24. To find a formula for $f(x)$ we will begin by finding the anti-derivative of the second derivative:

$$f''(x) = 4 - 6x - 40x^3,$$

to get

$$f'(x) = 4x - 3x^2 - 10x^4 + C.$$

To evaluate the constant C we can use the given value of the first derivative $f'(0) = 1$ to give:

$$1 = 4(0) - 3(0)^2 - 10(0)^4 + C$$

so that $C = 1$. To get the formula for $f(x)$ we will find the anti-derivative of the first derivative. Doing this gives:

$$f(x) = 2x^2 - x^3 - 2x^5 + x + C.$$

To find the value of this constant we can use the given function value of $f(0) = 2$. Substituting this into the formula for $f(x)$ given immediately above gives:

$$2 = 2(0)^2 - (0)^3 - 2(0)^5 + 0 + C$$

so that $C = 2$ and the final formula for $f(x)$ is: $f(x) = 2x^2 - x^3 - 2x^5 + x + 2$.