Solutions to Homework #8

Problems from Pages 203-205 (Section 4.1)

4. ***a*** Local maximum.

b Local minimum.

c Local minimum.

d Local minimum.

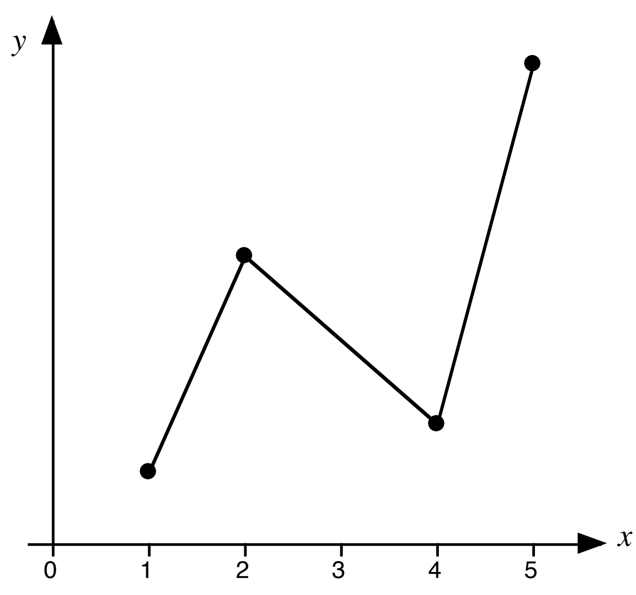
e Global (and local) maximum.

r Local minimum.

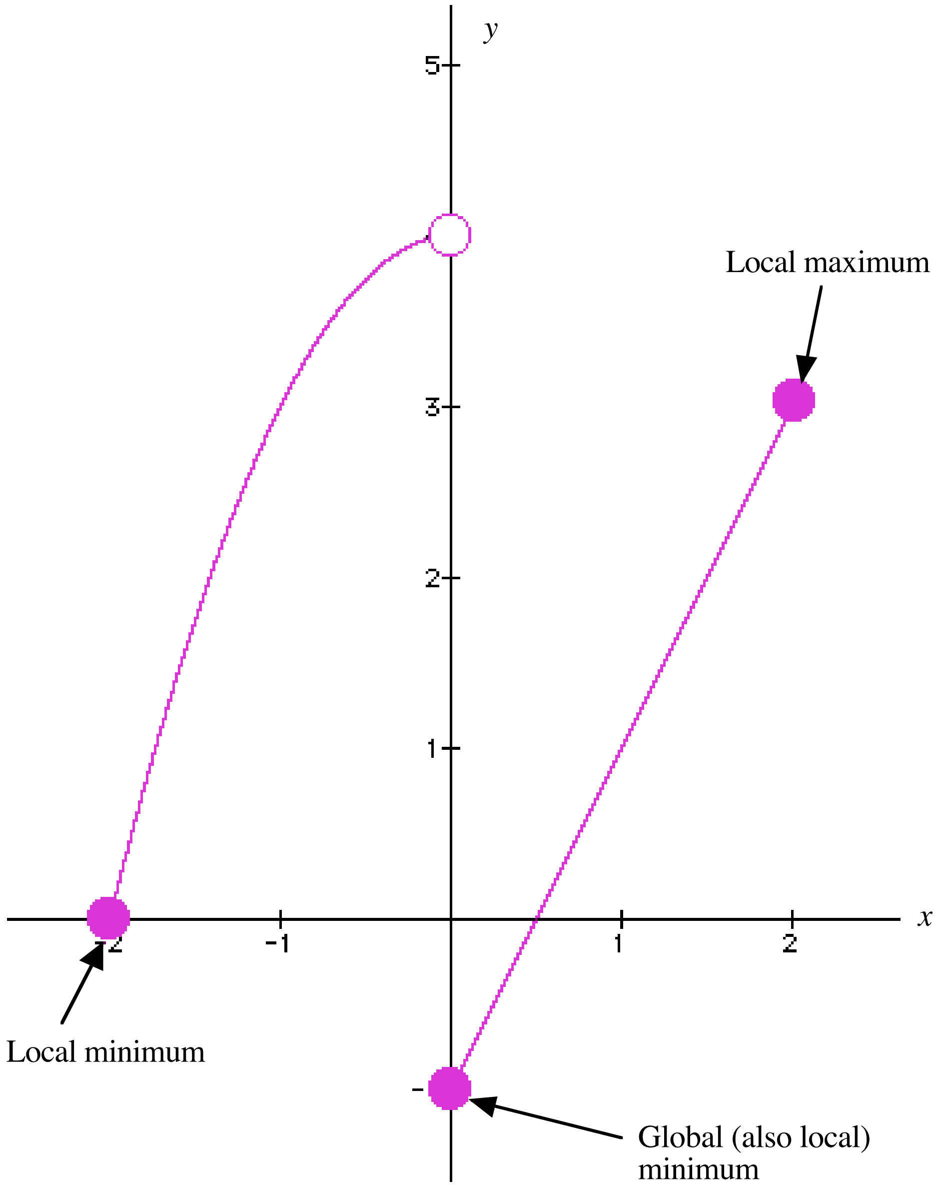
s Local maximum.

t Global (and local) minimum.

8. A picture of a graph with the necessary properties is shown below.



22. The graph of the piecewise defined function $f(x)$ is shown below.



The function $f(x)$ has a local maximum at the point (2, 3), a local minimum at the point (–2, 0) and its global maximum at the point (0, –1).

26. To find the critical numbers of $f(x)$ we need to find the x-coordinates of the points where the derivative is either zero or undefined. The first derivative of the function is:

$$f'(x) = 3x^2 + 2x + 1.$$

As the first derivative is defined for all values of *x*, the only critical numbers will be values of *x* where the derivative is equal to zero. These can be found using the quadratic formula:

$$x = \frac{-2 \pm \sqrt{2^2 - 4(3)(1)}}{6}.$$

As the discriminant is negative, the equation $f'(x) = 0$ does not have any solutions and the function *f*(*x*) does not have any critical numbers.

44. The absolute maximum and absolute minimum can occur at the following kinds of points:

(a) Points where the first derivative is equal to zero,

(b) Points where the first derivative is not defined, or,

(c) The endpoints of the interval [0, 3].

The first derivative of $f(x) = \frac{x}{x^2 + 4}$ is given by:

$$f'(x) = \frac{x^2 + 4 - 2x^2}{\left(x^2 + 4\right)^2} = \frac{-x^2 + 4}{\left(x^2 + 4\right)^2}.$$

The first derivative is always defined and equal to zero only at $x = \pm 2$. As we are only interested in the interval [0, 3], we need only focus on $x = 2$.

To find the global maximum and global minimum, we will evaluate *f*(*x*) at $x = 0$, $x = 2$ and $x = 3$. Doing this gives:

x	f(x)	Comments
0	0	Global minimum = 0
2	0.25	Global maximum = 0.25
3	$3/13 \approx 0.2307$	

58. Acceleration is the derivative of velocity, so if the velocity is given by the equation:

$$v(t) = 0.001302t^3 - 0.09029t^2 + 23.61t - 3.083,$$

then the acceleration will be given by:

$$a(t) = 0.003906t^2 - 0.18058t + 23.61.$$

We are interested in finding the global maximum and global minimum of $a(t)$ over the interval [0, 126].

The critical number of $a(t)$ is the solution of the equation:

$$a'(t) = 0.007812t - 0.18058 = 0,$$

which is $t = 23.11571941$. To find the global maximum and global minimum, we will evaluate $a(x)$ at $t = 0$, $t = 23.11571941$ and $t = 126$.

t	a(t)	Comments
0	23.61	
23.11571941	21.52288	Global minimum
126	62.86857	Global maximum

Problems from Pages 217-219 (Section 4.3)

8. **(a)** The intervals on which $f(x)$ is increasing or decreasing are the intervals on which the first derivative is positive or negative (respectively). The derivative of $f(x)$ is:

$$f'(x) = \ln(x) + 1.$$

The derivative is greater than zero when $\ln(x)$ is greater than -1. This occurs for the interval (e^{-1}, ∞). So, $f(x)$ is increasing on the interval (e^{-1}, ∞).

The function $f(x) = x \cdot \ln(x)$ is only defined for $x > 0$ so the interval on which the first derivative is negative will be $(0, e^{-1})$. The function $f(x)$ is decreasing on the interval $(0, e^{-1})$.

(b) The local maximum and local minimum values of $f(x)$ will occur at points where the derivative is zero or undefined. The only such point is $x = e^{-1}$. The second derivative of $f(x)$ is:

$$f''(x) = \frac{1}{x},$$

which is positive when $x = e^{-1}$ indicating that the point $(e^{-1}, -e^{-1})$ is a local minimum.

(c) The function $f(x)$ is concave up when the second derivative is positive and concave down when the second derivative is negative. As the second derivative is given by:

$$f''(x) = \frac{1}{x},$$

the second derivative is always positive for $x > 0$. This means that $f(x)$ is concave up over all of its domain and (as the concavity never changes) there are no inflection points.

10. The local maximums and local minimums of the function

$$f(x) = \frac{x}{x^2 + 4}$$

will occur at points where the derivative is either equal to zero or undefined. The derivative of $f(x)$ is:

$$f'(x) = \frac{x^2 + 4 - 2x^2}{\left(x^2 + 4\right)^2} = \frac{-x^2 + 4}{\left(x^2 + 4\right)^2}.$$

This derivative is never undefined so the local maximums and minimums will only occur at points where the derivative is equal to zero, which are the points with $x = -2$ and $x = +2$.

Classifying points using the First Derivative Test

x	–2.1	–2	–1.9
Derivative	–0.005796	0	+0.006734

As the first derivative goes from negative to positive, the point (–2, –0.25) is a local minimum.

x	1.9	2	2.1
Derivative	+0.006734	0	–0.005796

As the derivative goes from positive to negative, the point (2, 0.25) is a local maximum.

Classifying points using the Second Derivative Test

The second derivative of $f(x)$ is:

$$f''(x) = \frac{-2x\cdot\left(x^2 + 4\right)^2 - \left(-x^2 + 4\right)\cdot 2\left(x^2 + 4\right)\cdot 2x}{\left(x^2 + 4\right)^4} = \frac{-2x^3 - 8x + 4x^3 - 16x}{\left(x^2 + 4\right)^3} = \frac{2x^3 - 24x}{\left(x^2 + 4\right)^3}.$$

Evaluating this at $x = -2$ gives the second derivative value of 1, which is greater than zero, confirming that (–2, –0.25) is a local minimum.

Evaluating this at $x = 2$ give the second derivative a value of –0.5, which is less than zero, confirming that (2, 0.25) is a local maximum.

12. **(a)** The critical numbers of $f(x)$ are the x-values where the derivative is either zero or undefined. The derivative of $f(x)$ is:

$$f'(x) = 4x^3 \cdot (x-1)^3 + x^4 \cdot 3 \cdot (x-1)^2 = x^3 \cdot (x-1)^2 \cdot [7x-4].$$

The critical numbers of $f(x)$ are $x = 0$, $x = 1$ and $x = 4/7$.

(b) The second derivative of $f(x)$ is:

$$f''(x) = 3x^2 \cdot (x-1)^2 \cdot (7x-4) + x^3 \cdot 2(x-1) \cdot (7x-4) + x^3 \cdot (x-1)^2 \cdot 7.$$

Evaluating this at each of the three critical numbers gives the values listed in the table (below).

x	Second derivative
0	0
1	0
4/7	0.2399

The second derivative test tells us nothing about the behavior of $f(x)$ at the points (0, 0) or (1, 0). The second derivative test tells us that there is a local minimum at the point (4/7, –0.008393).

(c) We will carry out the First derivative test for each of the critical numbers and record the results in the tables shown below.

x	–0.1	0	0.1
Derivative	0.005687	0	–0.002673

x	0.9	1	1.1
Derivative	0.016767	0	0.049247

x	3/7	4/7	5/7
Derivative	–0.0257	0	0.029749

These results show that (0, 0) is a local maximum, (1, 0) is neither a local maximum nor a local minimum (it is a point of inflection) and (4/7, –0.008393) is a local minimum.

22. **(a)** The function $f(x)$ is increasing when the derivative is positive and decreasing when the derivative is negative. Between $x = 0$ and $x = 9$, the intervals on which the derivative is positive or negative are:

$f(x)$ is increasing: (1, 6) and (8, 9).

$f(x)$ is decreasing: (0, 1) and (6, 8).

(b) The function $f(x)$ will have a local minimum or local maximum at any points where the derivative is equal to zero. Between $x = 0$ and $x = 9$, the points at which the derivative is zero are:

$x = 1$ $x = 6$ $x = 8$.

(c) The function $f(x)$ is concave up when the derivative is increasing and concave down when the derivative is decreasing. Between $x = 0$ and $x = 9$, the intervals on which the derivative is increasing or decreasing are:

$f(x)$ is concave up: (0, 2), (3, 5) and (7, 9).

$f(x)$ is concave down: (2, 3) and (5, 7).

(d) The graph of $y = f(x)$ will display an inflection point at any x-coordinate of a point where the derivative function has a local maximum or local minimum. The x-coordinates of these points are:

$x = 2$, $x = 3$, $x = 5$ and $x = 7$.

(e) A sketch of the graph of $y = f(x)$ that passes through the point (0, 0) and that incorporates all of the information given above is shown below.