Solutions to Homework #7

Problems from Pages 173-175 (Section 3.4)

2. If we use $P(t)$ to represent the number of bacteria after t hours. As the population doubles every 20 minutes, $P(t)$ will be an exponential function with $P(0) = 60$ and $P(1/3) = 120$. The formula that will work in this situation is:

$$P(t) = 60 \cdot 8^t.$$

(a) The relative growth rate is equal to $\dfrac{P'(t)}{P(t)} = \dfrac{60 \cdot 8^t \cdot \ln(8)}{60 \cdot 8^t} = \ln(8)$.

(b) The formula for $P(t)$ is $P(t) = 60 \cdot 8^t$.

(c) The number of cells after 8 hours will be:

$$P(8) = 60 \cdot 8^8 = 1{,}006{,}632{,}960 \text{ cells.}$$

(d) The rate of growth after 8 cells will be the derivative evaluated when $t = 8$.

$$P'(8) = 60 \cdot 8^8 \cdot \ln(8) \approx 2.093 \text{ billion cells per hour.}$$

(e) To find the time when the number of cells reaches 20,000 we will solve the following equation for t:

$$20{,}000 = 60 \cdot 8^t.$$

$$t = \tfrac{1}{8} \cdot \ln\left(\tfrac{20000}{60}\right) \approx 2.79 \text{ hours.}$$

8. Let $M(t)$ be the number of milligrams of bismuth-210 that remain after t days. We are given that $M(0) = 800$ and $M(5) = 400$.

(a) The function $M(t)$ will be an exponential function $M(t) = A \cdot B^t$. To find the growth factor B we compute:

$$B = \left(\frac{400}{800}\right)^{\frac{1}{5-0}} \approx 0.8705505633.$$

The value of A is equal to $M(0) = 800$, so $M(t) = 800 \cdot (0.8705505633)^t$.

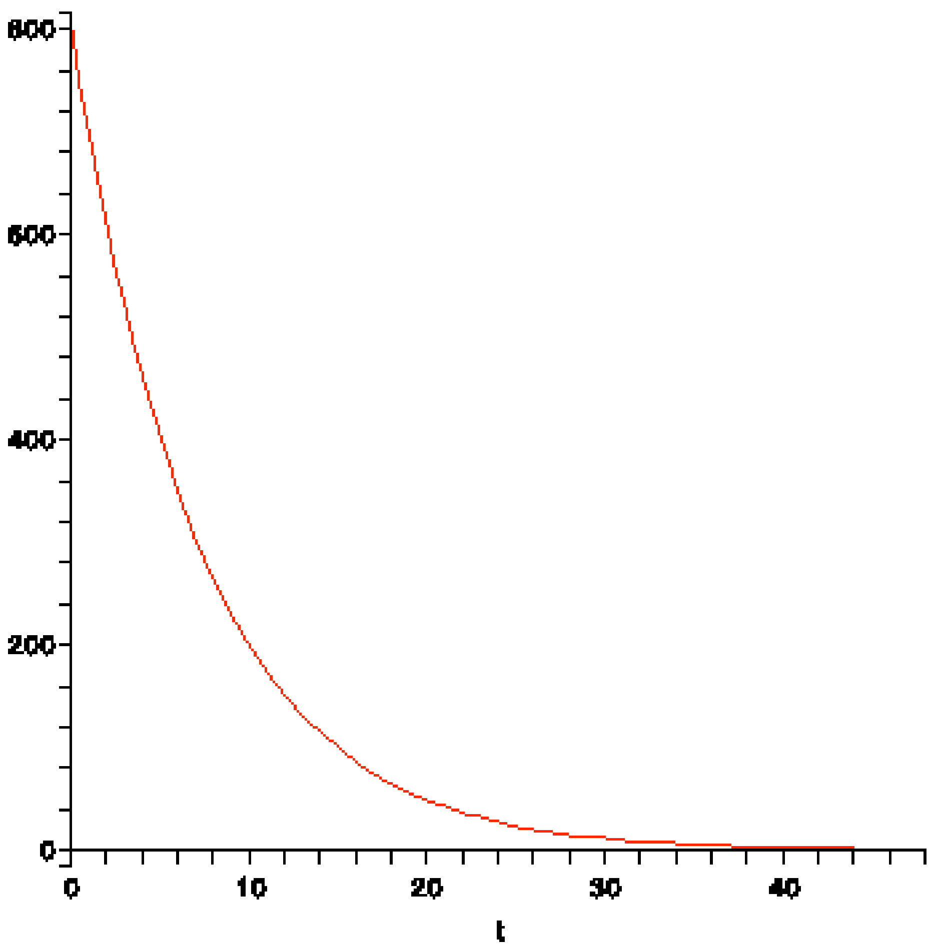
(b) The mass remaining after 30 days is $M(30) = 12.5$ mg.

(c) To find when the mass is reduced to 1 mg we will solve the following equation for t:

$$1 = 800\cdot(0.8705505633)^t.$$

$$t = \frac{\ln\left(\frac{1}{800}\right)}{\ln(0.8705505633)} \approx 48 \text{ days}.$$

(d) A graph showing $M(t)$ versus t is shown below.



16. We will use $H(t)$ to represent the temperature of the coffee (in ºC) t minutes after it is poured. As $H(t)$ obeys Newton's Law of Cooling, the formula for $H(t)$ will resemble:

$$H(t) = 20 + 75\cdot e^{-kt},$$

where k is a constant. We are told that when $H(t) = 70$, $H'(t) = -1$ °C/minute.

To determine when this happens, we need to solve the following equations for k and t:

$$H(t) = 20 + 75 \cdot e^{-kt} = 70$$

$$H'(t) = -k \cdot 75 \cdot e^{-kt} = -1.$$

From the first equation, $75 \cdot e^{-kt} = 50$. Substituting this into the second equation gives:

$$-50 \cdot k = -1$$

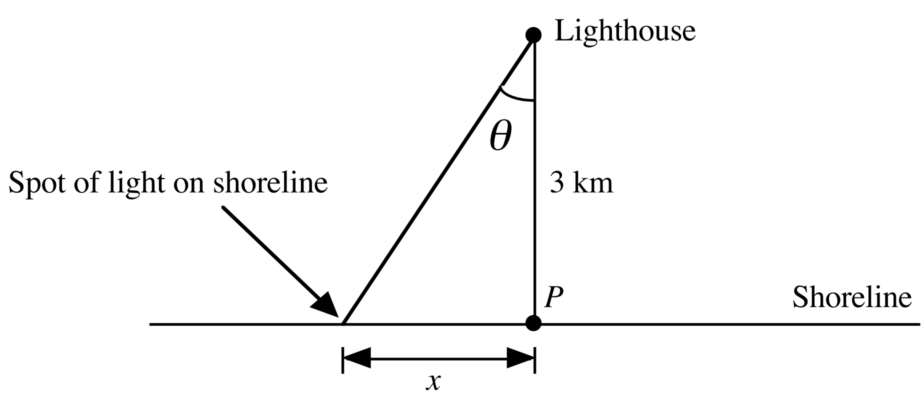
so that $k = 1/50$. Substituting $k = 1/50$ into the first equation and solving for t gives:

$$20 + 75 \cdot e^{-t/50} = 70$$

$$t = -50 \cdot \ln\left(\tfrac{50}{75}\right) \approx 20.27 \text{ minutes.}$$

Problems from Page 180 (Section 3.5)

40. The situation is illustrated (very schematically) below.



In this problem we want to calculate dx/dt when $x = 1$. We have that the rate of change of the angle θ with respect to time is $d\theta/dt = (4)(8\pi)$ radians per minute.

We will begin by finding a relationship between x and θ using trigonometry:

$$\tan(\theta) = \frac{x}{3},$$

or $x = 3 \cdot \tan(\theta)$. Taking the derivative of this and multiplying by $d\theta/dt$ gives:

$$\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt} = 3 \cdot \sec^2(\theta) \cdot 32\pi.$$

When $x = 1$, $\sec(\theta) \approx 1.054092553$ so that $dx/dt \approx 335.1032$ km/minute.

Problems from Page 186 (Section 3.6)

44. We begin with the function:

$$y = \frac{T}{\rho g} \cosh\left(\frac{\rho g x}{T}\right).$$

The plan to solve this problem will be to calculate each side of the equation:

$$\frac{d^2 y}{dx^2} = \frac{\rho g}{T} \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

and show that the two expressions we obtain are the same.

Now, everything in the above formula is a constant with the exception of x and y. Calculating the derivative of y with respect to x gives:

$$\frac{dy}{dx} = \frac{T}{\rho g} \sinh\left(\frac{\rho g x}{T}\right) \cdot \frac{\rho g}{T} = \sinh\left(\frac{\rho g x}{T}\right).$$

Calculating the second derivative gives:

$$\frac{d^2 y}{dx^2} = \cosh\left(\frac{\rho g x}{T}\right) \cdot \frac{\rho g}{T}.$$

For the right hand side of the equation:

$$\frac{\rho g}{T} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{\rho g}{T} \sqrt{1 + \sinh^2\left(\frac{\rho g x}{T}\right)} = \frac{\rho g}{T} \sqrt{\cosh^2\left(\frac{\rho g x}{T}\right)} = \frac{\rho g}{T} \cdot \cosh\left(\frac{\rho g x}{T}\right),$$

which is the same as the expression for the second derivative calculated above.

Problems from Pages 193-194 (Section 3.7)

4. As $x \to 0$, the limit

$$\lim_{x \to 0} \frac{x + \tan(x)}{\sin(x)}$$

is an indeterminant form of the 0/0 type. To evaluate this limit we can use L'Hopital's rule.

$$\lim_{x \to 0} \frac{x + \tan(x)}{\sin(x)} = \lim_{x \to 0} \frac{1 + \sec^2(x)}{\cos(x)} = \frac{1 + 1^2}{1} = 2.$$

10. As $x \to \infty$, the limit

$$\lim_{x \to \infty} \frac{\ln(\ln(x))}{x}$$

is an indeterminant form of the ∞/∞ type. To evaluate this limit we can use L'Hopital's rule.

$$\lim_{x \to \infty} \frac{\ln(\ln(x))}{x} = \lim_{x \to \infty} \frac{\frac{1}{\ln(x)} \cdot \frac{1}{x}}{1} = \lim_{x \to \infty} \frac{1}{x \cdot \ln(x)} = 0.$$

16. As $x \to 0$, the limit

$$\lim_{x \to 0} \frac{\cos(mx) - \cos(nx)}{x^2}$$

is an indeterminant form of the 0/0 type. To evaluate this limit we can use L'Hopital's rule.

$$\lim_{x \to 0} \frac{\cos(mx) - \cos(nx)}{x^2} = \lim_{x \to 0} \frac{-m \cdot \sin(mx) + n \cdot \sin(nx)}{2x}.$$

This is also an indeterminant form of the 0/0 type. To evaluate this limit we can use L'Hopital's rule a second time.

$$\lim_{x \to 0} \frac{\cos(mx) - \cos(nx)}{x^2} = \lim_{x \to 0} \frac{-m^2 \cdot \cos(mx) + n^2 \cdot \cos(nx)}{2} = \frac{n^2 - m^2}{2}.$$

30. Before calculating the limit of the function we will combine the two fractions into one.

$$\frac{1}{\ln(x)} - \frac{1}{x-1} = \frac{x-1-\ln(x)}{(x-1)\cdot\ln(x)}.$$

As $x \to 1$ this is an indeterminant form form of the 0/0 type. To evaluate this limit we can use L'Hopital's rule.

$$\lim_{x\to 1}\left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right) = \lim_{x\to 1}\frac{x-1-\ln(x)}{(x-1)\cdot\ln(x)} = \lim_{x\to 1}\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln(x)} = \lim_{x\to 1}\frac{x-1}{x-1+x\cdot\ln(x)}.$$

As $x \to 1$ this is also an indeterminant form form of the 0/0 type. To evaluate this limit we can use L'Hopital's rule a second time.

$$\lim_{x\to 1}\left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right) = \lim_{x\to 1}\frac{x-1-\ln(x)}{(x-1)\cdot\ln(x)} = \lim_{x\to 1}\frac{1}{1+1+\ln(x)} = \frac{1}{1+1} = \frac{1}{2}.$$

40. As $x \to \infty$, the limit

$$\lim_{x\to\infty}\frac{\ln(x)}{x^p}$$

is an indeterminant form of the ∞/∞ type. To evaluate this limit we can use L'Hopital's rule.

$$\lim_{x\to\infty}\frac{\ln(x)}{x^p} = \lim_{x\to\infty}\frac{\frac{1}{x}}{p\cdot x^{p-1}} = \lim_{x\to\infty}\frac{1}{p\cdot x^p} = 0.$$