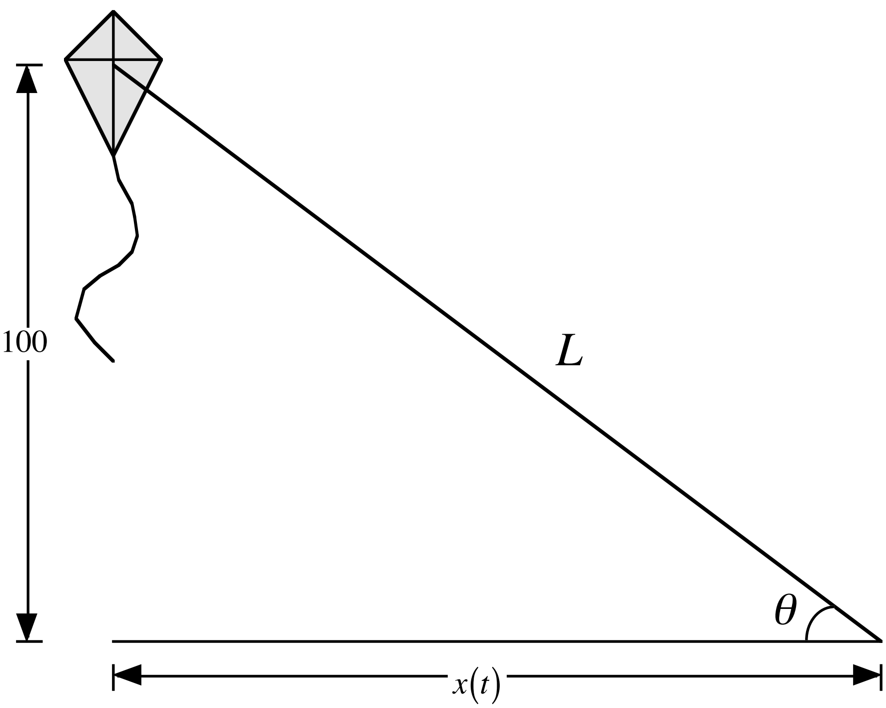
Solutions to Homework #6

Problems from Pages 131-133 (Section 2.7)

26. The situation described on page 132 is illustrated below.



In this problem we are given the derivative $x'(t) = 8$ feet per second. We are asked to find the value of the derivative of $\theta(t)$ when $L = 200$. To do this we must first find a relationship between the quantities $x(t)$ and $\theta(t)$.

The relationship between the quantities $x(t)$ and $\theta(t)$ is given by the trigonometric formula:

$$\tan(\theta(t)) = \frac{100}{x(t)}.$$

Taking the derivative of each side of this equation with respect to t using the Chain Rule gives:

$$\sec^2(\theta(t)) \cdot \theta'(t) = \frac{-100 \cdot x'(t)}{x(t)^2} \qquad . \qquad . \qquad . \qquad (1).$$

At the moment of time when $L = 200$, the Theorem of Pythagoras gives that:

$$x(t) = \sqrt{L^2 - 100^2} = \sqrt{200^2 - 100^2} = 173.205 \text{ feet.}$$

Next noting that the secant of an angle is equal to the ratio of the hypotenuse over the adjacent side gives that at the moment when $L = 200$ feet:

$$\sec(\theta(t)) = \frac{L}{x(t)} = \frac{200}{173.205}.$$

Substituting these values and the derivative of $x(t)$ into the Equation (1) and solving for the derivative of $\theta(t)$ gives the rate at which the angle is decreasing:

$$\theta'(t) = \frac{-1}{50} \text{ radians per second.}$$

Problems from Page 147 (Section 3.1)

18. The objective of this problem is to find a formula for an exponential function,

$$f(x) = A \cdot B^x,$$

given the two points (0, 2) and (2, 2/9). First we will calculate the growth factor, B. Doing this:

$$B = \left(\frac{\frac{2}{9}}{2}\right)^{\frac{1}{2-0}} = \frac{1}{3}.$$

To find the value of A we will plug one of the points (say (0, 2)) into the exponential equation. Doing this gives:

$$2 = A \cdot \left(\tfrac{1}{3}\right)^0 = A.$$

Putting all of this together gives: $f(x) = 2 \cdot \left(\tfrac{1}{3}\right)^x$.

26. To find the value of the limit:

$$\lim_{x \to \infty} \frac{2 + 10^x}{3 - 10^x},$$

we will begin by dividing all of the terms in both numerator and denominator by 10^x so that we gain terms whose limits can be found as $x \to \infty$. Doing this gives:

$$\lim_{x \to \infty} \frac{\frac{2}{10^x} + \frac{10^x}{10^x}}{\frac{3}{10^x} - \frac{10^x}{10^x}} = \lim_{x \to \infty} \frac{\frac{2}{10^x} + 1}{\frac{3}{10^x} - 1} = \frac{0+1}{0-1} = -1.$$

Problems from Pages 158-160 (Section 3.2)

20. To find the inverse of the function:

$$m = f(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

we will rearrange this equation to make m the subject of the equation. Note that we will assume that $v \geq 0$ to guarantee that this function passes the Horizontal Test.

$$\sqrt{1 - \frac{v^2}{c^2}} = \frac{m_0}{m}$$

$$1 - \frac{v^2}{c^2} = \left(\frac{m_0}{m}\right)^2$$

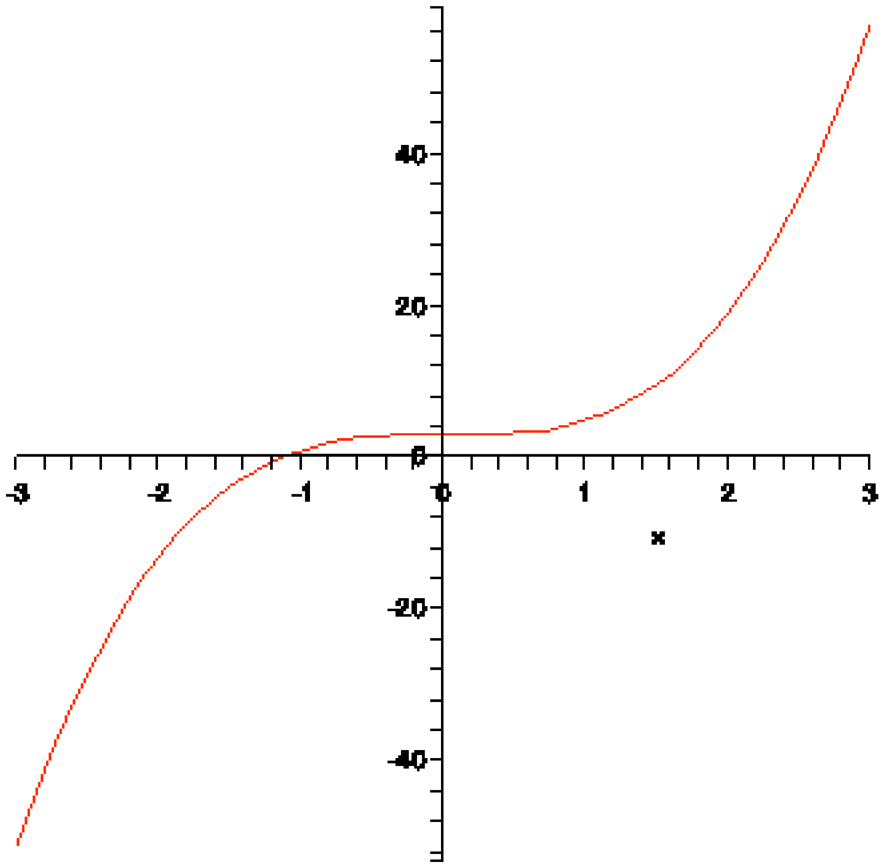
$$\frac{v^2}{c^2} = 1 - \left(\frac{m_0}{m}\right)^2$$

$$v^2 = c^2 \cdot \left(1 - \left(\frac{m_0}{m}\right)^2\right)$$

$$v = f^{-1}(m) = c \cdot \sqrt{1 - \left(\frac{m_0}{m}\right)^2}.$$

This inverse takes the dilated mass m of the particle as its input and gives the velocity required to achieve this dilation as its output.

22. To find a formula for the inverse of the function $y = 2x^3 + 3$ we will start by verifying that the inverse exists using the Horizontal Line test. The graph of the function is shown on the next page. Any horizontal line that you care to draw on this graph will only touch the graph in one place, so the Horizontal Line test is passed and the function will have an inverse that has a formula we can actually find.



Next we will rearrange the equation $y = 2x^3 + 3$ to make x the subject of the equation.

$$2x^3 = y - 3$$

$$x^3 = \frac{y-3}{2}$$

$$x = f^{-1}(y) = \sqrt[3]{\frac{y-3}{2}}.$$

38. To find the derivative of the inverse of the function:

$$f(x) = \sqrt{x^3 + x^2 + x + 1}$$

at the point $a = 2$, we first note that $f(1) = 2$, so that $f^{1}(2) = 1$. Next, the derivative of the function $f(x)$ is given by:

$$f'(x) = \frac{3x^2 + 2x + 1}{2\sqrt{x^3 + x^2 + x + 1}}$$

so that:

$$f'\left(f^{-1}(2)\right) = f'(1) = \frac{3+2+1}{2\sqrt{1+1+1+1}} = \frac{3}{2}.$$

The derivative of the inverse at $a = 2$ is then given by:

$$\left(f^{-1}\right)'(2) = \frac{1}{f'\left(f^{-1}(2)\right)} = \frac{2}{3}.$$

62. **(a)** To solve the equation $e^{2x+3} - 7 = 0$:

$$e^{2x+3} = 7$$

$$2x + 3 = \ln(7)$$

$$2x = \ln(7) - 3$$

$$x = \frac{\ln(7) - 3}{2} \approx -0.5270449255.$$

(b) To solve the equation $\ln(5 - 2x) = -3$:

$$5 - 2x = e^{-3}$$

$$2x = 5 - e^{-3}$$

$$x = \frac{5 - e^{-3}}{2} = 2.475106466.$$

Problems from Pages 166-167 (Section 3.3)

28. The derivative of the function $f(x) = \dfrac{1 - xe^x}{x + e^x}$ can be found using the quotient and product rules. The derivative is:

$$f'(x) = \frac{-x^2e^x - e^{2x} - e^x - 1}{\left(x + e^x\right)^2}.$$

42. To find an equation for the tangent line to the curve $y = e^x/x$ at the point $(1, e)$ we will begin by finding a formula for the derivative:

$$\frac{dy}{dx} = \frac{e^x(x-1)}{x^2}.$$

To find the slope of the tangent line we will evaluate the derivative at $x = 1$ giving that the slope of the tangent line is zero. The tangent line is the horizontal line that passes through the point $(1, e)$, the equation of which is $y = e$.

60. **(a)** To find the limit of the function

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

as $t \to \infty$, note that as $k > 0$ the limit of ae^{-kt} as $t \to \infty$ is zero. Therefore:

$$\lim_{t \to \infty} \frac{1}{1 + ae^{-kt}} = \frac{1}{1+0} = 1.$$

(b) The rate of spread of the rumor is the derivative of $p(t)$. Differentiating this function using the quotient rule and the chain rule gives:

$$p'(t) = \frac{kae^{-kt}}{\left(1 + ae^{-kt}\right)^2}.$$

(c) The graph of $p(t)$ with $a = 10$ and $k = 0.5$ is shown below. Also shown on this graph is a horizontal line with a height of 0.8. The t-coordinate of the point where this horizontal line intersects the graph of $p(t)$ is the estimate of how long it takes for 80% of the population to hear the rumor. The intersection point appears to occur at approximately $t = 7.5$, so it will take about 7.5 hours for the rumor to spread to 80% of the population.