Solutions to Homework #5

Problems from Pages 119-121 (Section 2.5)

50. The values of $f(x)$, $g(x)$ and their derivatives are supplied by the table on Page 120 of the textbook.

(a) Since $F(x) = f(f(x))$, applying the Chain Rule gives:

$$F'(2) = f'(f(2)) \cdot f'(2) = f'(1) \cdot f'(2) = (4)(5) = 20.$$

(b) Since $G(x) = g(g(x))$, applying the Chain Rule gives:

$$G'(3) = g'(g(3)) \cdot g'(3) = g'(2) \cdot g'(3) = (7)(9) = 63.$$

52. The values of $f(x)$ and its derivatives are estimated from the graph on Page 120.

(a) Since $h(x) = f(f(x))$, applying the Chain Rule gives:

$$h'(2) = f'(f(2)) \cdot f'(2) = f'(1) \cdot f'(2) \approx \left(\tfrac{-3}{4}\right)(-1) = \tfrac{3}{4}.$$

(b) Since $g(x) = f(x^2)$, applying the Chain Rule gives: $g'(x) = f'(x^2) \cdot 2x$. Plugging in $x = 2$ and the estimates from the graph gives:

$$g'(2) = f'(4) \cdot 4 \approx (2)(4) = 8.$$

64. **(a)** The derivative dV/dr is the instantaneous rate of change of the volume with respect to the radius of the balloon. The derivative dV/dt is the instantaneous rate of change of the volume with respect to time.

(b) The formula for the volume of a sphere, V, as a function of the radius of the sphere, r, is:

$$V = \frac{4\pi}{3} r^3.$$

Taking the derivative of this gives $\frac{dV}{dr} = 4\pi r^2$. However this is not the derivative of volume that we want. We want dV/dt. If we assume that r is a function of time, then we can write the Chain Rule using Newton's notation. It reads:

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}.$$

Problems from Pages 125-127 (Section 2.6)

6. We will begin by making the substitution $y = f(x)$ in the given equation.

$$\left(f(x)\right)^5 + x^2 \cdot \left(f(x)\right)^3 = 1 + x^4 \cdot f(x).$$

Next we will take the derivative of each term.

$$5 \cdot f(x)^4 \cdot f'(x) + 2x \cdot f(x)^3 + 3 \cdot x^2 \cdot f(x)^2 \cdot f'(x) = 4x^3 \cdot f(x) + x^4 \cdot f'(x).$$

Next we will rearrange to get every term that includes the derivative as a factor on the left hand side of the equation and every other term on the right hand side of the equation.

$$5 \cdot f(x)^4 \cdot f'(x) + 3 \cdot x^2 \cdot f(x)^2 \cdot f'(x) - x^4 \cdot f'(x) = 4x^3 \cdot f(x) - 2x \cdot f(x)^3$$

To make the derivative the subject of the equation we will factor out the derivative and then divide.

$$f'(x) \cdot \left[5 \cdot f(x)^4 + 3 \cdot x^2 \cdot f(x)^2 - x^4\right] = 4x^3 \cdot f(x) - 2x \cdot f(x)^3$$

$$f'(x) = \frac{4x^3 \cdot f(x) - 2x \cdot f(x)^3}{5 \cdot f(x)^4 + 3 \cdot x^2 \cdot f(x)^2 - x^4}.$$

Finally, to restore the result to the notation used in the problem as it was posed in the textbook, we will substitute for $f(x)$ and its derivative.

$$\frac{dy}{dx} = \frac{4x^3 \cdot y - 2x \cdot y^3}{5 \cdot y^4 + 3 \cdot x^2 \cdot y^2 - x^4}.$$

10. We will begin by making the substitution $y = f(x)$ in the given equation.

$$f(x) \cdot \sin\left(x^2\right) = x \cdot \sin\left(f(x)^2\right).$$

Next we will take the derivative of each term.

$$f'(x) \cdot \sin\left(x^2\right) + f(x) \cdot \cos\left(x^2\right) \cdot 2x = \sin\left(f(x)^2\right) + x \cdot \cos\left(f(x)^2\right) \cdot 2 \cdot f(x) \cdot f'(x).$$

Next we will rearrange to get every term that includes the derivative as a factor on the left hand side of the equation and every other term on the right hand side of the equation.

$$f'(x)\cdot\sin\left(x^2\right)-x\cdot\cos\left(f(x)^2\right)\cdot 2\cdot f(x)\cdot f'(x)=\sin\left(f(x)^2\right)-f(x)\cdot\cos\left(x^2\right)\cdot 2x.$$

To make the derivative the subject of the equation we will factor out the derivative and then divide.

$$f'(x)\cdot\left[\sin\left(x^2\right)-x\cdot\cos\left(f(x)^2\right)\cdot 2\cdot f(x)\right]=\sin\left(f(x)^2\right)-f(x)\cdot\cos\left(x^2\right)\cdot 2x$$

$$f'(x)=\frac{\sin\left(f(x)^2\right)-f(x)\cdot\cos\left(x^2\right)\cdot 2x}{\sin\left(x^2\right)-x\cdot\cos\left(f(x)^2\right)\cdot 2\cdot f(x)}.$$

Finally, to restore the result to the notation used in the problem as it was posed in the textbook, we will substitute for $f(x)$ and its derivative.

$$\frac{dy}{dx}=\frac{\sin\left(y^2\right)-y\cdot\cos\left(x^2\right)\cdot 2x}{\sin\left(x^2\right)-x\cdot\cos\left(y^2\right)\cdot 2y}.$$

18. To find the equation of the tangent line we need to know the derivative evaluated at the point where $x = 1$ and $y = 2$. To do this we will use implicit differentiation, and follow the same steps used in the solutions of Questions 6 and 10 (above).

$$x^2+2xy-y^2+x=2$$

$$x^2+2x\cdot f(x)-f(x)^2+x=2$$

$$2x+2\cdot f(x)+2x\cdot f'(x)-2f(x)\cdot f'(x)+1=0$$

$$2x\cdot f'(x)-2f(x)\cdot f'(x)=-1-2x-2\cdot f(x)$$

$$f'(x)\cdot\left[2x-2f(x)\right]=-1-2x-2\cdot f(x)$$

$$f'(x)=\frac{-1-2x-2\cdot f(x)}{2x-2f(x)}$$

$$\frac{dy}{dx} = \frac{-1 - 2x - 2y}{2x - 2y}.$$

To get the slope of the tangent line we will plug $x = 1$ and $y = 2$ into the formula for the derivative. Doing this gives:

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(1,2)} = \frac{-1-2-4}{2-4} = \frac{7}{2}.$$

As the point of tangency has coordinates $(x, y) = (1, 2)$ we can write down the equation of the tangent line in point-slope form:

$$y - 2 = \frac{7}{2}(x - 1).$$

44. Our objective in this problem is to find the height of the lamp, which we will represent by the symbol h. To do this, we will find an equation for the tangent line that touches the top of the ellipse and then use this tangent line equation to find h.

To find the equation of the tangent line to the ellipse we first need the derivative for the ellipse.

The equation of the ellipse is $x^2 + 4y^2 = 5$. Using implicit differentiation to find the derivative of y with respect to x gives (after all of the usual steps have been carried out):

$$\frac{dy}{dx} = \frac{-x}{4y}.$$

We do not know the x and y-coordinates of the point where the tangent line touches the ellipse. So that we can refer to them easily, we will write $x = a$ and $y = b$ for the point where the tangent line touches the top of the ellipse. Plugging these symbols into the equation for the derivative gives one expression for the slope of the tangent line:

$$m = \frac{-a}{4b}.$$

We can see from the diagram that the tangent line passes through the point (a, b) and the point $(-5, 0)$. Calculating the change in y over the change in x for these two points will give us another expression for the slope of the tangent line.

$$m = \frac{b-0}{a-(-5)} = \frac{b}{a+5}.$$

Setting the two expressions for the slope of the tangent line equal to each other gives that:

$$\frac{-a}{4b} = \frac{b}{a+5} \qquad \text{or} \qquad 4b^2 = -a^2 - 5a.$$

Since the point (a, b) is on the ellipse, it obeys the equation for the ellipse so that:

$$5 = 4b^2 + a^2 = -5a$$

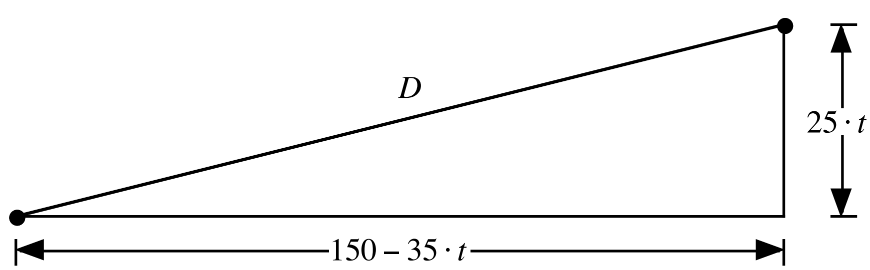
and $a = -1$. Plugging this into the equation for the ellipse and solving for b gives $b = 1$ and the slope of the tangent line is 0.25. The point (–5, 0) is on the tangent line, so we can write down the equation of the tangent line in point-slope form. It is:

$$y = 0.25(x + 5).$$

The light is located at $x = 3$. Plugging this into the equation for the tangent line gives $y = 2$, so that the height of the lamp, h, is $h = 2$.

Problems from Pages 131-133 (Section 2.7)

10. The positions of the two ships t hours after noon are shown in the diagram given below.



Using the Theorem of Pythagoras, the distance between the two ships t hours after noon will be give by:

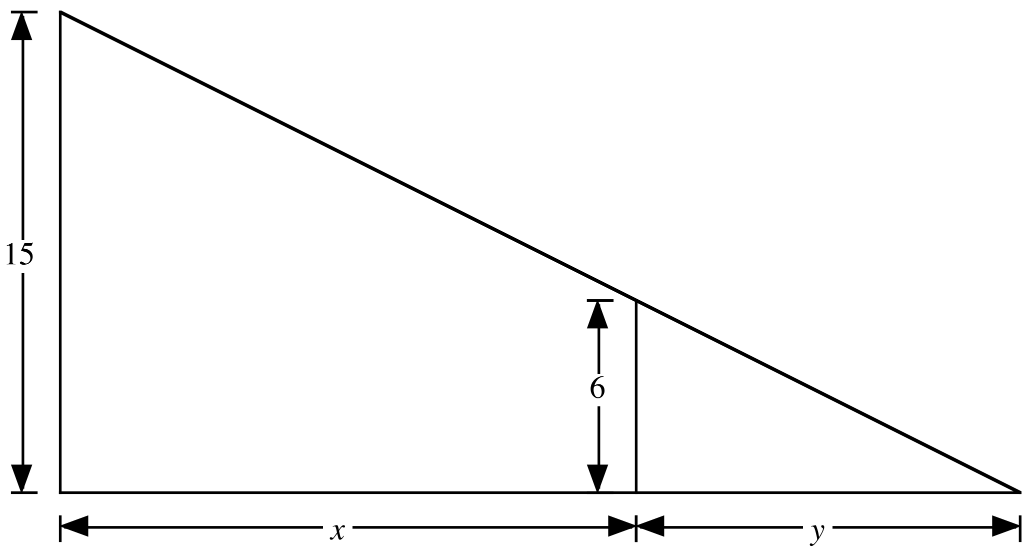
$$D = \sqrt{(25t)^2 + (150-35t)^2} = \sqrt{22500 - 10500t + 1850t^2}.$$

To find the rate at which this distance is changing at 4:00pm ($t = 4$) we can calculate the derivative and plug in $t = 4$. Doing this:

$$D'(t) = 0.5\left(22500 - 10500t + 1850t^2\right)^{-0.5} \cdot \left(-10500 + 3700 \cdot t\right)$$

$$D'(4) = 0.5\left(22500 - 10500(4) + 1850(4)^2\right)^{-0.5} \cdot \left(-10500 + 3700(4)\right) = 21.39 \text{ km/hour.}$$

12. The situation is represented abstractly in the diagram shown below. In this diagram, y is the length of the man's shadow (in feet) and x is the distance between the man and the lamppost (also in feet). The hypotenuse of the triangle is formed by the ray of light from the lamp that just goes over the man's head.



The derivative we are given is $\frac{dx}{dt} = 5$ feet per second. The derivative that we want is $\frac{dx}{dt} + \frac{dy}{dt}$. We can find a relationship between x and y using the Principle of Similar Triangles. Based on the diagram given above, this gives that:

$$\frac{y}{6} = \frac{x+y}{15}, \text{ so that:}$$

$$y = \frac{10}{15}x = \frac{2}{3}x.$$

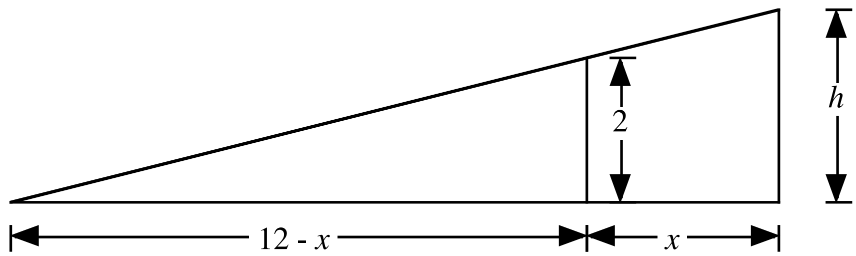
Taking derivatives of both sides of this equation with respect to time and plugging in the value for dx/dt that we were given leads to:

$$\frac{dy}{dt} = \frac{2}{3} \cdot \frac{dx}{dt} = \frac{10}{3} \text{ feet per second.}$$

Adding this to the given value for dx/dt gives that the tip of the shadow moves at:

$$\frac{dx}{dt} + \frac{dy}{dt} = 5 + \frac{10}{3} = \frac{25}{3} \text{ feet per second.}$$

14. The situation is summarized abstractly in the diagram shown below. The symbol x is used to represent the distance of the man to the building and the symbol h is used to represent the height of the man's shadow.



The derivative that we are given is $dx/dt = -1.6$ meters per second. The derivative that we want is dh/dt when $x = 4$.

We will again use the Principle of Similar Triangles to find a relationship between the quantities x and h. Applying similar triangles to the diagram given above yields:

$$\frac{h}{12} = \frac{2}{12 - x}.$$

We will multiply each side of this equation by 12, and then take derivatives with respect to time, t. Doing this gives:

$$\frac{dh}{dt} = \frac{24}{(12 - x)^2} \cdot \frac{dx}{dt}.$$

Plugging $x = 4$ and $dx/dt = -1.6$ into this gives:

$$\frac{dh}{dt} = \frac{24}{(12 - 4)^2} \cdot (-1.6) = -0.6 \text{ meters per second.}$$

So, when the man is 4 meters from the building, the height of his shadow is decreasing by 0.6 meters per second.