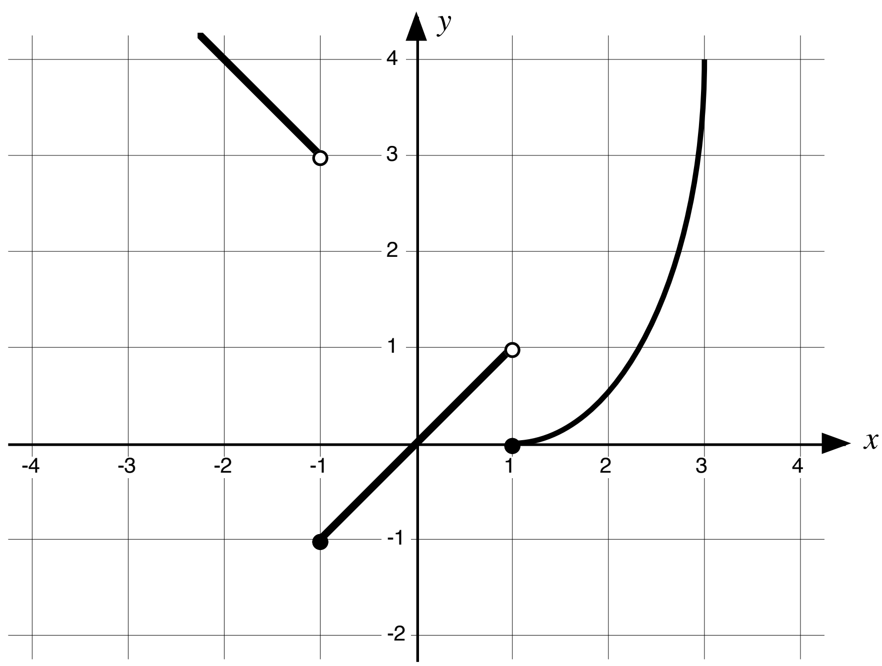
Solutions to Homework #3

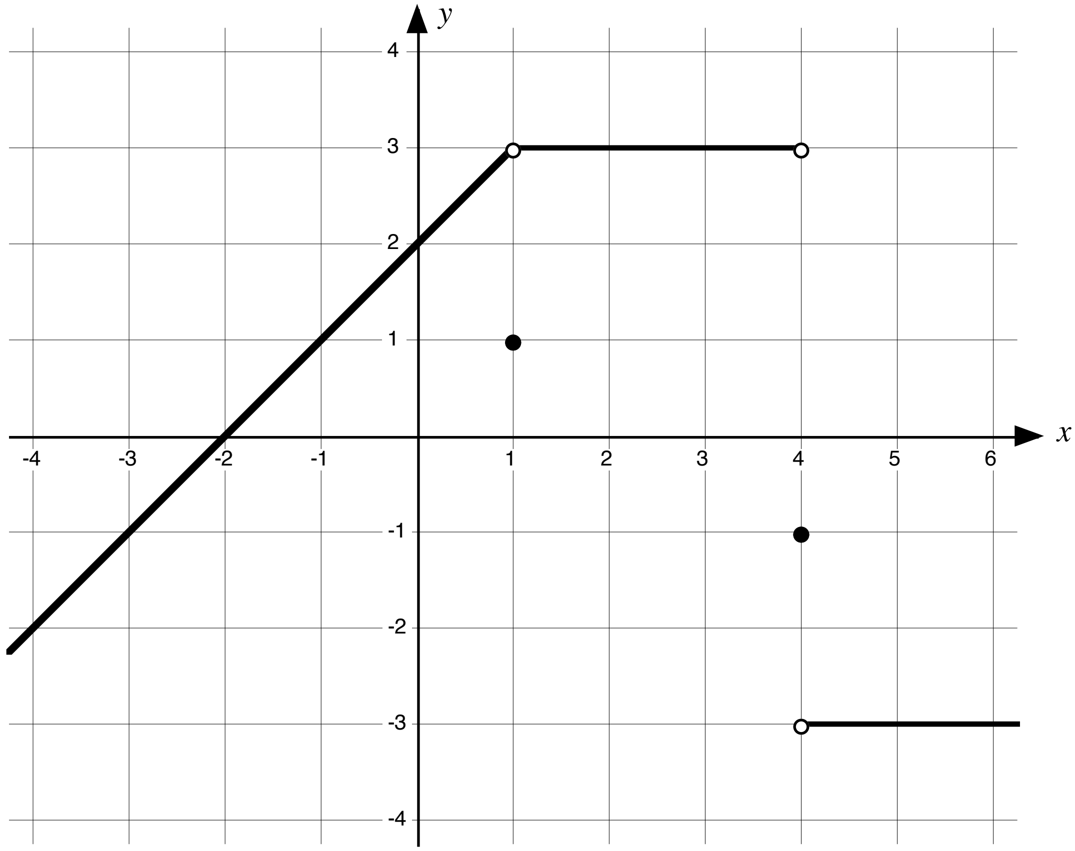
Problems from Pages 33-35 (Section 1.3)

6. The graph of the function $f(x)$ that was defined in pieces is shown below.



The places where the left-hand and right-hand limits will not agree are at the points $x = -1$ and $x = 1$. The two values of a for which the limit of $f(x)$ will not exist as $x \to a$ are therefore $a = -1$ and $a = 1$.

10. There are many, many different graphs that could correctly answer this question. The one shown below is not the only acceptable answer.



12. The results of plugging the given x values into the rational function:

$$f(x) = \frac{x^2 - 2x}{x^2 - x - 2}$$

are summarized in the table shown below.

x	f(x)
0	0
–0.5	–1
–0.9	–9
–0.95	–19
–0.99	–99
–0.999	–999
–1.001	1001
–1.01	101
–1.1	11
–1.5	3
–2	2

Based on the values in the table, I would guess that the limit of $f(x)$ as $x \to -1$ does not exist. This is because the values seem to get larger (but also more negative) as we approach −1 from the left but get large and positive as we approach negative one from the right. It seems unlikely that the left and right hand limits will agree when we get to $x = -1$.

Problems from Pages 43-45 (Section 1.4)

2. The values of the limits that exist (and the explanations for those that do not exist) are given below. In each case, the value was determined by reading y-values from the appropriate graph as x approached the value in question.

(a) $\lim_{x \to 2} \left[f(x) + g(x)\right] = 2 + 0 = 0$.

(b) $\lim_{x \to 1} \left[f(x) + g(x)\right]$ does not exist. This is because as x approaches 1 from the left, the graph of $y = g(x)$ approaches a y-value of $y = 2$. However, as x approaches 1 from the right, the graph of $y = g(x)$ approaches a y-value of $y = 1$. As these two y-values are not equal, the limit of $g(x)$ (as $x \to 1$) does not exist, so any expression that includes this limit will not exist, either.

(c) $\lim_{x \to 0} \left[f(x) \cdot g(x)\right] = 0$.

(d) $\lim_{x \to -1} \dfrac{f(x)}{g(x)}$ does not exist. Near $x = -1$, the value of $f(x)$ is also close to −1. However, near $x = -1$ the value of $g(x)$ is close to zero, being small and negative to the left of $x = -1$ and small and positive to the right of $x = -1$. When we form the quotient of $f(x/g(x)$ for values of x that are slightly to the left of $x = -1$, we will have a number that is close to −1 over a small negative number. The value of the quotient will be a large positive number. On the right of $x = -1$, the quotient will consist of a number close to −1 over a small negative number, giving the quotient a large negative value overall. As the quotient approaches different values from the left and right as $x \to -1$, the limit of the quotient does not exist.

(e) $\lim_{x \to 2} \left[x^3 \cdot f(x)\right] = 2^3 \cdot 2 = 16$.

(f) $\lim_{x \to 0} \sqrt{3 + f(x)} = \sqrt{3 + 1} = 2$.

10. **(a)** One way to express the problem with the given equation:

$$\frac{x^2 + x - 6}{x - 2} = x + 3,$$

is to note that the domains of the functions on each side of the equation are different. The domain of the function on the left hand side of the equation consists of all real numbers except for $x = 2$, whereas the domain of the function on the right hand side of the equation consists of all real numbers.

(b) The limit equation:

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} (x + 3)$$

is correct because when we compute this limit we are not interested in what the behavior of either function is at the exact point $x = 2$. Instead, we are interested in what each function does when x is near but not actually at the point $x = 2$ and from this we infer what "should" happen at $x = 2$ if the nearby trend in the graph continued there. The limits of both functions as $x \to \infty$ are equal because the behavior of $\frac{x^2 + x - 6}{x - 2}$ and $(x + 3)$ are identical near (but not at) $x = 2$.

14. The point $x = 4$ is not in the domain of the rational function:

$$f(x) = \frac{x \cdot (x-4)}{(x+1) \cdot (x-4)},$$

so finding the limit of this function as $x \to 4$ is not as easy as just plugging $x = 4$ into the rational function's formula. However, when x is near 4 (but not actually equal to 4), it is true that:

$$\frac{x \cdot (x-4)}{(x+1) \cdot (x-4)} = \frac{x}{x+1} \qquad \text{provided } x \neq 4.$$

The limit that we have to calculate will be the same as the limit of this simpler rational function as $x \to 4$, which is:

$$\lim_{x \to 4} \frac{x^2 - 4x}{x^2 - 3x - 4} = \lim_{x \to 4} \frac{x}{x+1} = \frac{4}{5}.$$

22. Probably the biggest obstacle to calculating this limit is the use of the "–1" exponents, which actually denote reciprocals (hence fractions). Recognizing this and then combining the fractions gives the following expression:

$$\frac{(3+h)^{-1}-3^{-1}}{h} = \frac{\frac{1}{3+h}-\frac{1}{3}}{h} = \frac{\frac{3-(3+h)}{(3+h)\cdot 3}}{h} = \frac{\frac{-h}{(3+h)\cdot 3}}{h} = \frac{-1}{(3+h)\cdot 3},$$

where the last simplification is only valid when $h \neq 0$. Taking the limit of the above gives:

$$\lim_{h \to 0} \frac{(3+h)^{-1}-3^{-1}}{h} = \lim_{h \to 0} \frac{-1}{(3+h)\cdot 3} = \frac{-1}{9}.$$

Problems from Pages 66-69 (Section 1.6)

2. The values of each of the limits are given below.

(a) $\lim_{x \to \infty} g(x) = 2$. This assumes that as x gets larger and larger the graph gets closer and closer to the horizontal asymptote located at $y = 2$.

(b) $\lim_{x \to -\infty} g(x) = -2$. This assumes that as x takes on larger and larger negative values the graph gets closer and closer to the horizontal asymptote located at $y = -2$.

(c) $\lim_{x \to 3} g(x) = +\infty$. This is because the graph rises (without any apparent upper bounds) as x approaches 3 from both left and right.

(d) $\lim_{x \to 0} g(x) = -\infty$. This is because the graph drops (without any apparent lower bounds) to larger and larger negative y-values as x approaches 3 from both left and right.

(e) $\lim_{x \to 2^+} g(x) \approx 0.8$. As x approaches 2 from the right, the graph appears to approach a height that is slightly less than 1.

(f) The vertical asymptotes are located at: $x = -2$, $x = 0$ and $x = 3$. The horizontal asymptotes are located at $y = -2$ and $y = 2$.

22. There are many valid ways to do this calculation. One way to approach it is to observer that when $x \geq -2$,

$$x + 2 = \sqrt{(x+2)^2}\text{, so that:}$$

$$\frac{x+2}{\sqrt{9x^2+1}} = \frac{\sqrt{(x+2)^2}}{\sqrt{9x^2+1}} = \sqrt{\frac{(x+2)^2}{9x^2+1}} = \sqrt{\frac{x^2+4x+4}{9x^2+1}}.$$

Paying attention to the inside of the square root, we can see that the 'Dominator of the Denominator" is $9x^2$. Dividing all of the terms within the square root by this gives:

$$\frac{x+2}{\sqrt{9x^2+1}} = \sqrt{\frac{\frac{x^2}{9x^2} + \frac{4x}{9x^2} + \frac{2}{9x^2}}{\frac{9x^2}{9x^2} + \frac{1}{9x^2}}} = \sqrt{\frac{\frac{1}{9} + \frac{4}{9x} + \frac{2}{9x^2}}{1 + \frac{1}{9x^2}}}.$$

Taking the limit of these terms as $x \to \infty$ then gives that:

$$\lim_{x \to \infty} \frac{x+2}{\sqrt{9x^2+1}} = \sqrt{\frac{\frac{1}{9}+0+0}{1+0}} = \sqrt{\frac{1}{9}} = \frac{1}{3}.$$