SOLUTIONS

Recitation Handout 9: Optimizing the Design for Bottled Water

Today's recitation will focus on optimization problems. These are problems where you are try to find the maximum or minimum value of a quantity (usually subject to a constraint).

For each of the problems you will attempt on today's recitation, you will be trying to minimize the surface area of a water bottle with the constraint that the volume of the water bottle has to equal a certain fixed value.

The specific goals of today's recitation are for you to:

1. Use geometrical formulas for perimeters, areas, volumes and surface areas to create functions for surface areas and volumes.

2. Locate the critical points of a function by calculating the first derivative of a function f and then solving the equation:
$$f'(x) = 0.$$

3. Recall the relationship between the behavior of a derivative and the concavity of the original function (see table below).

Second Derivative is ...	Derivative is ...	Original function is ...
Positive	Increasing	Concave up
Negative	Decreasing	Concave down
Zero	Neither increasing nor decreasing	Possibly at a point of inflection

4. Use the change in sign of the first derivative evaluated near a critical point to classify the critical point as a local maximum or local minimum.

5. Use the sign of the second derivative evaluated at a critical point to classify the critical point as a local maximum or a local minimum.

The aquapod may be the most environmentally friendly water bottle as its shape is close to a sphere.

The overall goal in each of the optimization problems you will complete during today's recitation will be to find the most environmentally friendly dimensions for each common way of designing a plastic water bottle. The amount of plastic used to manufacture the bottle is proportional to the surface area of the bottle, which is why we try to minimize the surface area each time.

However, the surface area formulas that you set up will normally involve at least two variables. You will also typically set up an equation for the volume of the bottle. This equation (sometimes called the *constraint*) will allow you to eliminate one of the variables from your surface area formula.

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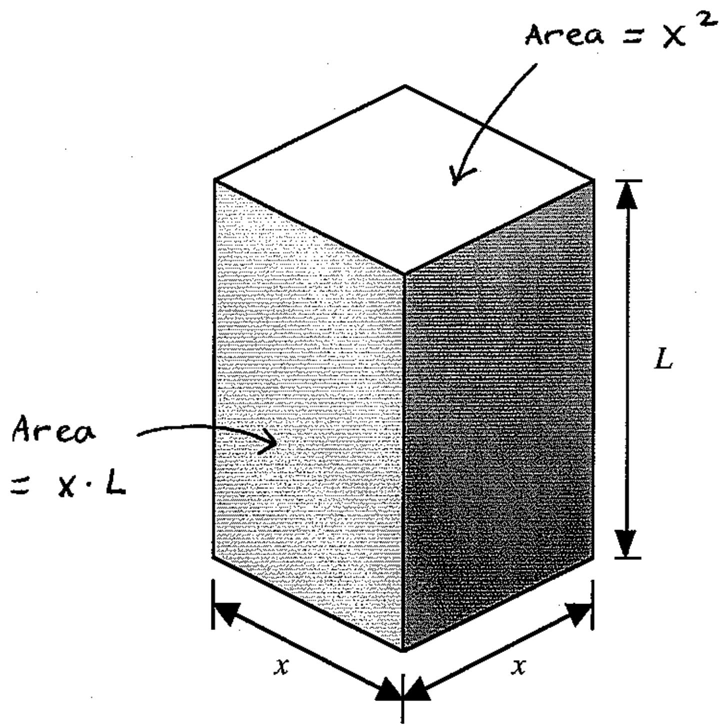
How environmentally friendly is a FIJI brand plastic water bottle?

Fiji brand bottled water comes in a bottle that closely resembles a rectangular prism with a square base (see the picture). These bottles contain one point five liters of water (which is the same as 1500 cubic centimeters).

1. The amount of plastic that is used to create the water bottle will be most closely related to the surface area of the bottle.

Because the actual bottle slopes at the top, this is actually pretty hard to calculate. As a first attempt at this problem, we will use the surface area of the simplified version of the water bottle that is shown below.

Find a formula (involving x, L and numbers) for the surface area of the water bottle.



Surface area $= 2x^2 + 4x \cdot L$

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2. The total volume of a bottle of FIJI brand water is 1500 cubic centimeters. Express this fact as an equation involving x, L and numbers.

$$x^2 \cdot L = 1500$$

3. Use your answers to Questions 1 and 2 to create a formula for the surface area of the bottle that involves only the variable x and numbers.

$$L = \frac{1500}{x^2}.$$

Plug this into the formula for surface area to eliminate L:

$$\text{Surface area} = S = 2x^2 + 4x\left(\frac{1500}{x^2}\right)$$

$$= 2x^2 + \frac{6000}{x}$$

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4. Find the value of x that makes the surface area as small as possible.

$$\frac{dS}{dx} = 4x - \frac{6000}{x^2} = 0$$

$$4x^3 = 6000$$

$$x^3 = 1500$$

$$x = (1500)^{1/3} \approx 11.4471 \text{ cm}$$

5. Find the value of L that makes the surface area as small as possible.

$$L = \frac{1500}{(11.4471)^2} \approx 11.4471 \text{ cm}$$

6. How do you know that the values of x and L that you found actually give the *smallest* surface area possible? Use the first derivative of surface area to confirm that this is the case.

x	11.4	11.4471	11.5
First derivative	-0.568	0	0.631

$-$ $\rightarrow$ $+$

The negative to positive sign change in the first derivative shows that $x = (1500)^{1/3}$ gives a local minimum for surface area.

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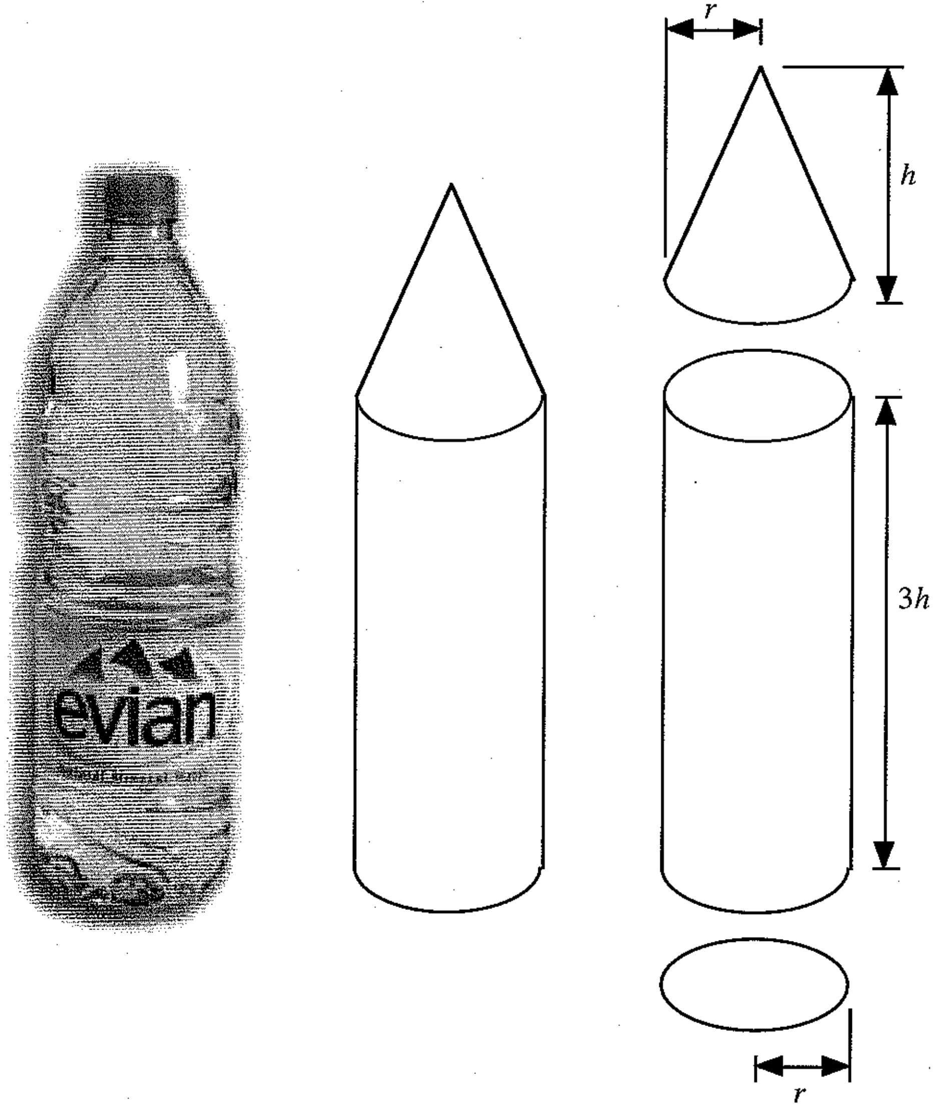
How environmentally friendly is an Evian brand water bottle?

One point five liter (1500 cubic centimeters) bottles of Evian mineral water come in a bottle that consists of a cone and a cylinder.



7. Find a formula (in terms of r and h) for the volume of the Evian water bottle.

$$\text{Volume} = V = \frac{1}{3}\pi r^2 \cdot h + \pi r^2 (3h)$$

$$= \frac{10}{3}\pi r^2 \cdot h.$$

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8. Find a formula (in terms of r and h) for the surface area of the Evian water bottle. Remember (as we learned in class) that the surface area of a cone with height h and radius r is given by:

$$S = \pi r \sqrt{r^2 + h^2}.$$

$$\text{Surface area} = S = \pi r \cdot \sqrt{r^2 + h^2} + 2\pi r \cdot (3h) + \pi r^2$$

$$= \pi \cdot r \sqrt{r^2 + h^2} + 6\pi r h + \pi r^2$$

9. Use the volume formula that you found to eliminate h from the surface area formula.

$$V = \frac{10}{3} \pi r^2 \cdot h = 1500$$

Rearrange to make h the subject of this equation:

$$h = \frac{1500}{\frac{10}{3} \pi r^2} = \frac{450}{\pi r^2}$$

Substitute this into the formula for surface area:

$$S = \pi r \cdot \sqrt{r^2 + \left(\frac{450}{\pi r^2}\right)^2} + 6\pi r \left(\frac{450}{\pi r^2}\right) + \pi r^2$$

$$= \pi \cdot \sqrt{r^4 + \frac{450^2}{\pi^2 r^2}} + \frac{2700}{r} + \pi r^2$$

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10. What is the value of r that gives the smallest possible value for the surface area of the bottle?

$$\frac{dS}{dr} = \frac{\pi\left(4r^3 - \frac{3(450)^2}{\pi^2 r^3}\right)}{2\cdot\sqrt{r^4 + \frac{450^2}{\pi^2 r^2}}} - \frac{2700}{r^2} + 2\pi r = 0$$

This is a very difficult equation to solve for r. Using a graphing calculator to solve the equation gives:

$$r \doteq 6.3598248$$

11. What is the value of h that gives the smallest possible value for the surface area of the bottle?

$$h = \frac{450}{\pi(6.3598)^2} \approx 3.54138 \text{ cm}$$

12. How do you know that the values of r and h that you found actually give the *smallest* surface area possible? Use the second derivative of surface area to confirm that this is the case.

$$\frac{d^2S}{dr^2} = -\frac{\pi}{4}\left(r^4 + \frac{450^2}{\pi^2 r^2}\right)^{-3/2}\cdot\left(4r^3 - \frac{3(450)^2}{\pi^2 r^3}\right)^2$$

$$+ \frac{\pi\left(12r^2 + \frac{9(450)^2}{\pi^2 r^4}\right)}{2\sqrt{r^4 + \frac{450^2}{\pi^2 r^2}}} + \frac{5400}{r^3} + 2\pi$$

Plugging $r = 6.3598$ into this gives $\left.\frac{d^2S}{dr^2}\right|_{r=6.3598} > 0.$

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Summary

13. Examine a FIJI and an Evian water bottle. Which seems like the most wasteful?

I suspect the FIJI bottle is more wasteful. This is because round shapes usually do a better job of containing a given volume with the smallest possible surface area.

14. According to the values of x and L that you calculated, what is the smallest possible surface area for a FIJI brand water bottle?

$$\text{Surface area} \approx 6 \cdot (11.4471)^2 = 786.2165 \text{ cm}^2$$

15. Use a ruler to measure the actual dimensions of a FIJI brand water bottle. What is the actual surface area of the bottle?

From the actual bottle, $x \doteq 8$ cm and $L \doteq 22$ cm.

Surface area ≈ 832 cm^2.

16. By what percentage does the amount of plastic in the FIJI bottle exceed the minimum?

$$\% = \frac{832 - 786}{786} \times \frac{100}{1} \approx 5.8\ \%$$

17. According to the values of r and h that you calculated, what is the smallest possible surface area for an Evian water bottle?

Surface area ≈ 697.036 cm^2.

18. Use a ruler to measure the actual dimensions of an Evian water bottle. What is the actual surface area of the bottle?

From the actual bottle, $r \approx 4$ cm and $h \approx 9$ cm.

Surface $\doteq 838$ cm^2.

19. By what percentage does the amount of plastic in the Evian bottle exceed the minimum?

$$\% = \frac{838 - 697}{697} \times \frac{100}{1} \doteq 20.2\ \%$$

20. Which of the two bottles (Evian or FIJI) is actually the mast wasteful?

Evian appears to be more wasteful.