SOLUTIONS

Recitation Handout 8: Introduction to Constrained Optimization

Today's recitation will focus on optimization problems. These are problems where you are trying to locate the maximum or minimum value(s) of a function. Sometimes you consider all possible x-values and sometimes you consider only a specific interval of x-values when searching for the maximum (or minimum) value of the function.

The specific goals for today's recitation are for you to be able to:

1. Locate the critical points of a function by calculating the first derivative of a function f and then solving the equation: $f'(x) = 0$.

2. Use the sign of the first derivative on either side of a critical point to classify the point as a local maximum, local minimum or neither.

3. Recognize when the problem domain of a function does not include all real numbers (i.e. when the problem domain is restricted to a particular interval).

4. Locate the global maximum and global minimum of a function whose domain is a closed interval.

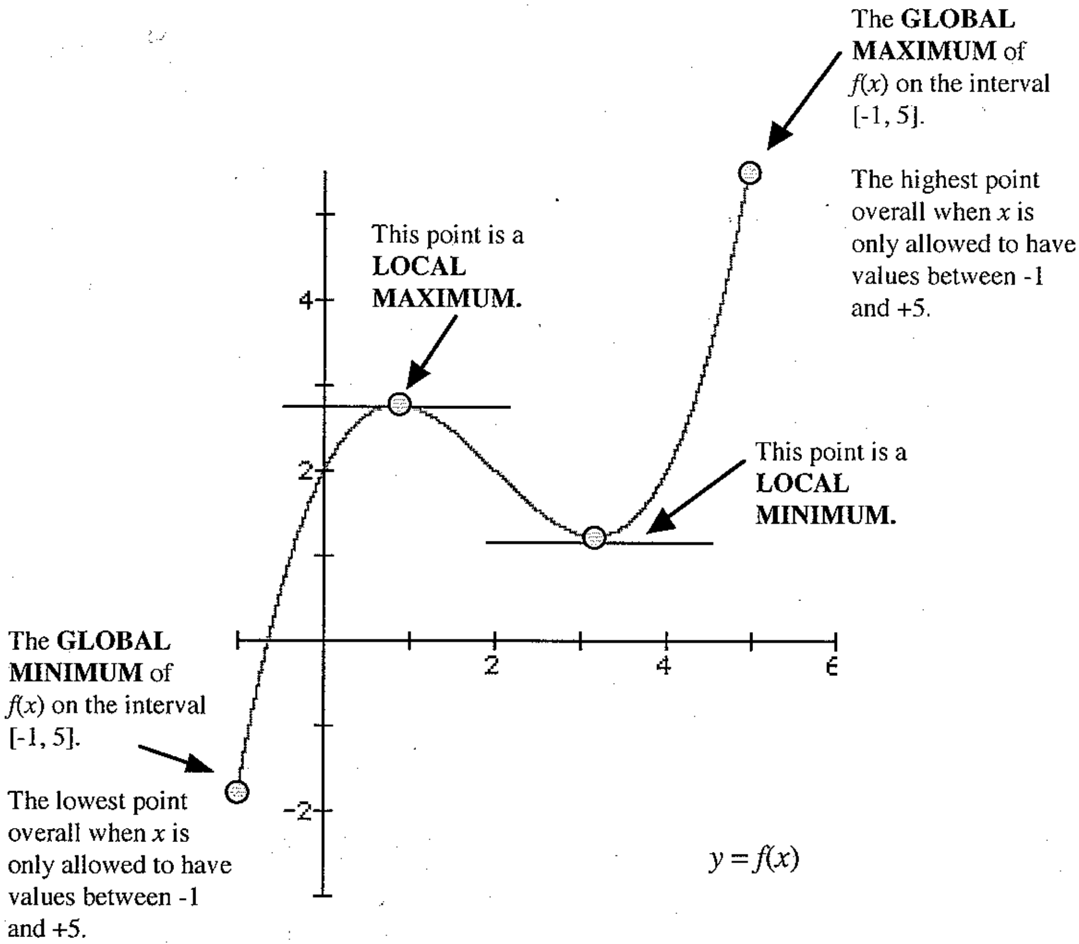
5. Use formulas for perimeters, areas, volumes and surface areas to create functions to model situations that are described graphically, verbally, and/or numerically.

Local and Global Maximums and Minimums

The **global maximum** of a function is the highest height that the function ever reaches. The **global minimum** of a function is the lowest height that the function ever reaches.

Not every maximum or minimum point is a global maximum or global minimum. These lesser maximums and minimums are called **local maximums** and **local minimums** because they are higher or lower than the point immediately surrounding them, but are not the highest of lowest possible.

SOLUTIONS



A Step-by-Step Procedure for Finding Maximums and Minimums

The following outline may help to guide your work when you work on maximum and minimum problems.

Step 1: If the situation has been described graphically or verbally, try to find an equation to represent the function that you want to maximize/minimize as well as equations to represent any constraints or conditions that have to be satisfied.

Step 2: Differentiate the function and use the derivative to locate the critical points.

Step 3: Once you have located the critical points, use the derivative to decide if they are maximums, minimums or neither.

Step 4: If the domain of the function is a closed interval, check the value of the functions at the end-points and compare to the value of the function at the critical points.

SOLUTIONS

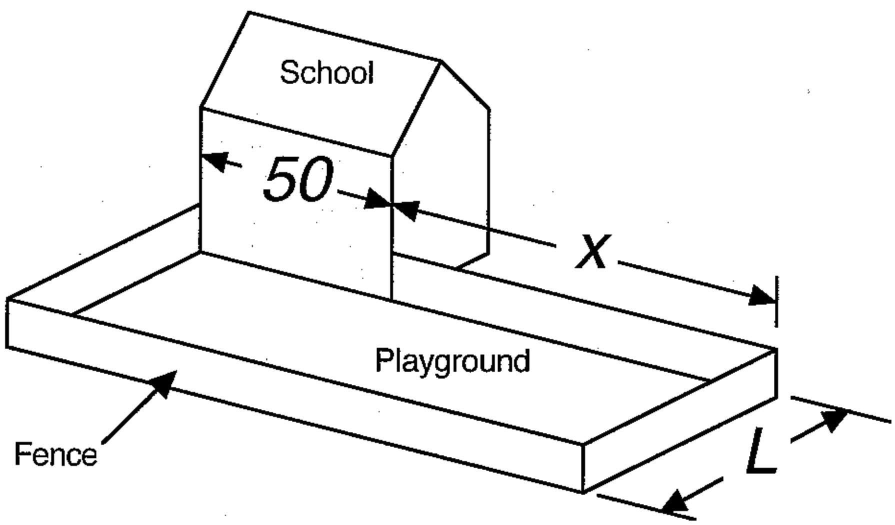
Step 5: If there are any points at which the derivative will be undefined (e.g. sharp points caused by absolute values or functions defined in pieces), check the value of the function at these points too.

Step 6: Look back. Do the locations and values of the maximum/minimum seem reasonable?

The step that most people find difficult about optimization problems is setting up the function to optimize in the first place (i.e. Step 1). We will spend as much time as possible in class working through lots of optimization problems so that we can gain experience in the area of setting up the equations to optimize.

I Remember the Days in the Old School Yard

Due to an outbreak of nefarious activity on school property, the school board of a small town decides to build a fence around part of the school's playground. The side of the schoolhouse is 50 feet long and will form part of the fence.



1. Write down a formula for the area of the playground that will be enclosed by the fence. Your formula may include the variables x and L.

$$\text{Area} = (x + 50) \cdot L$$

SOLUTIONS

2. After soliciting bids, the school board decides that they can afford 250 feet of fence. Express this fact as an equation involving x, L and numbers.

$$2L + 2x + 50 = 250$$

3. Use your answers to Questions 1 and 2 to create a formula for the area of the playground enclosed by the fence that includes *only* the variable x and numbers.

$$2L + 2x = 200$$

$$L = 100 - x.$$

Substituting this into the area formula gives:

$$\text{Area} = (x + 50)(100 - x)$$

$$= 5000 + 50x - x^2$$

SOLUTIONS

4. Find the value of x that will allow the largest possible area to be enclosed by the fence.

$$\frac{d\ \text{Area}}{dx} = 50 - 2x = 0$$

$$x = 25 \text{ feet.}$$

5. Find the value of L that will allow the largest possible area to be enclosed by the fence.

$$L = 100 - 25 = 75 \text{ feet.}$$

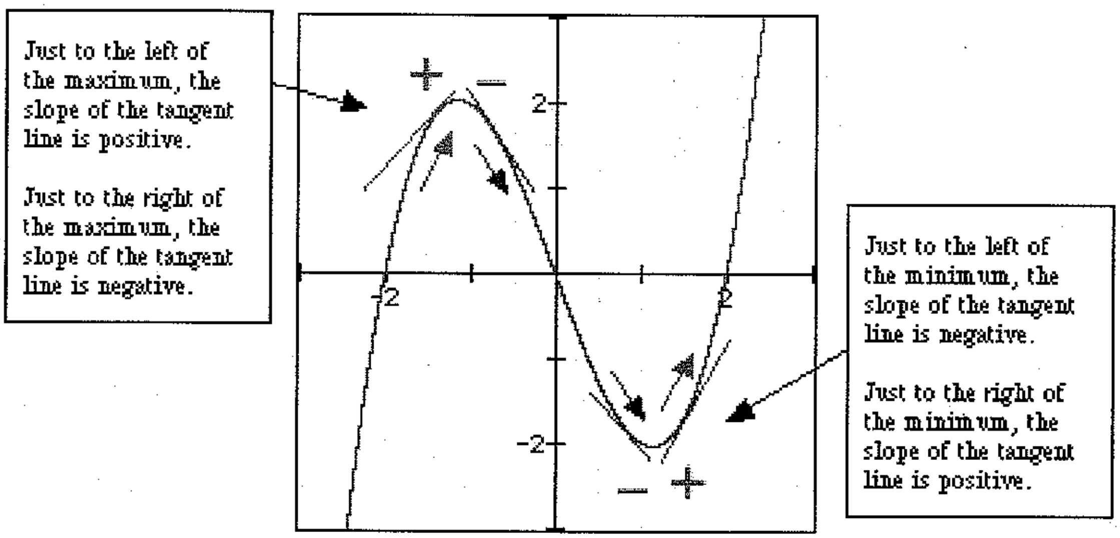
SOLUTIONS

Classifying Critical Points

Sometimes when you are trying to find the maximum or minimum values of a function there will be more than one place where the derivative of the function is equal to zero. How can you tell maximum points from minimum points? One way is to check the sign of the derivative on each side of the critical point.

Sign of derivative just to the left of the critical point	Sign of derivative just to the right of the critical point	Type of critical point
+	−	Maximum
−	+	Minimum

The reason that examining the signs of the derivative on either side of a critical point can tell you about the nature of the point (i.e. whether it is a maximum or minimum) is indicated below.



Example

Find the x-coordinates of the points where the derivative of the function:

$$f(x) = x^3 - 5x + 1$$

is equal to zero. Classify each point as a local minimum or local maximum.

SOLUTIONS

Solution

Step 1: Find the derivative

$$f'(x) = 3 \cdot x^2 - 5$$

Step 2: Set the derivative equal to zero and solve for x

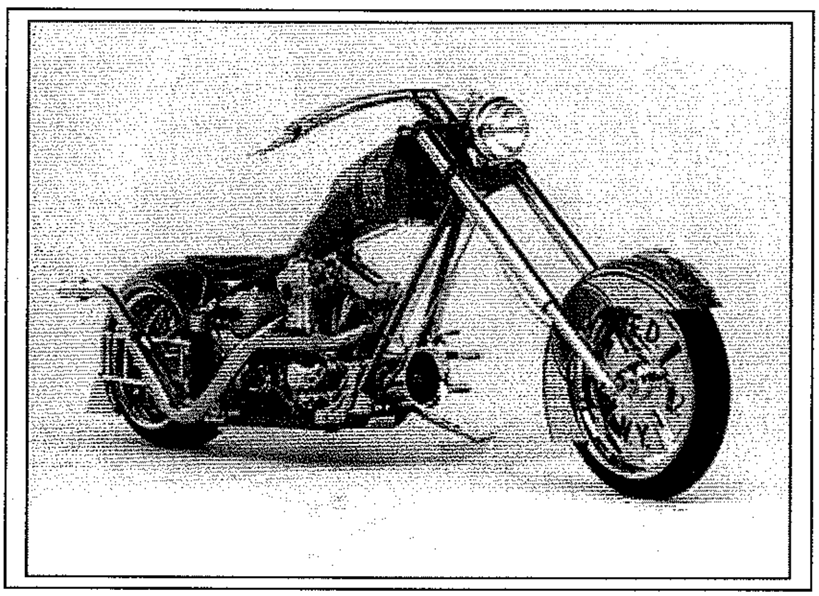
$$3 \cdot x^2 - 5 = 0$$

$$x = \pm\sqrt{\frac{5}{3}} \approx \pm 1.29$$

Step 3: Use the sign of the derivative to determine which point is a maximum and which point is a minimum

x	$f'(x) = 3 \cdot x^2 - 5$	Interpretation
−1.3	+0.07	X=−1.29 is a maximum
−1.2	−0.68	
1.2	−0.68	X=1.29 is a minimum
1.3	+0.07	

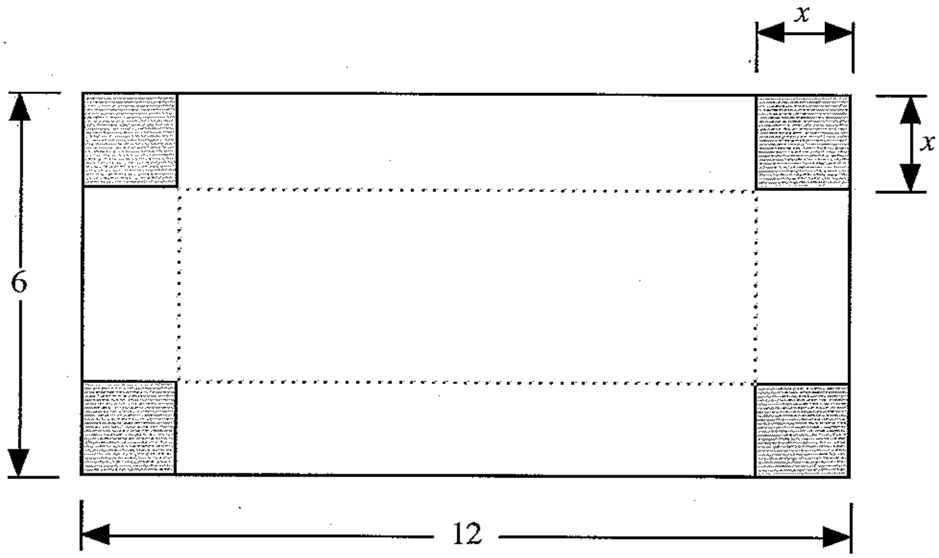
A Tool Box for the Fire Bike



On the hit TV show *American Chopper*, Paul Teutul and his son Paul Teutul, Jr. build custom motorcycles. During an early episode, Paul Teutul, Jr. built a motorcycle styled after a fire truck. During the construction process, his father helped out by making a toolbox out of sheet metal to attach to the bike.

Paul Teutul formed the toolbox by starting with a sheet of metal that measured 12 inches long by 6 inches wide. He cut squares out of the corners and then folded up the sides to create the box. (See the diagram on the next page for his cutting plan.)

SOLUTIONS



6. Use the diagram shown above to find a formula for the volume of the box when it is assembled. The only variable in your formula should be x.

$$V = x(6-2x)(12-2x)$$
$$= x(72 - 36x + 4x^2)$$
$$= 72x - 36x^2 + 4x^3$$

7. Find the x-coordinates of all points where the derivative of volume is zero.

$$\frac{dV}{dx} = 72 - 72x + 12x^2 = 0.$$

To find x, use the quadratic equation:

$$x = \frac{72 \pm \sqrt{(72)^2 - (4)(12)(72)}}{(2)(12)}$$

$$= 1.268 \quad , \quad 4.732 \quad \text{inches.}$$

SOLUTIONS

8. Classify the points that you found in Question 7 as maximum or minimum values of the volume.

$x = 1.268$

x	$\frac{dV}{dx}$
1.26	0.3312
1.28	-0.4992

Local maximum.

$x = 4.732$

x	$\frac{dV}{dx}$
4.7	-1.32
4.8	2.88

Local minimum.

10. What is the smallest value that x could realistically have? What is the largest value that x could realistically have? What value(s) do you get for the volume when you plug these x-values into your volume formula from Question 6?

$x = 0$ is the smallest (although it wouldn't be much of a box).

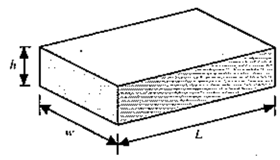
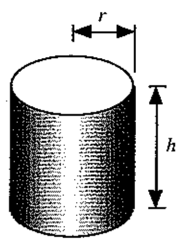
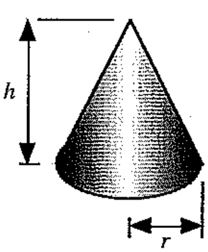
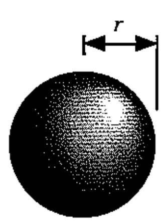
$x = 3$ is the largest.

11. If Paul Teutl, Sr. wanted to build a box that had the largest possible volume, what dimensions would the finished box have?

1.268 inches by 3.464 inches by 9.464 inches.

SOLUTIONS

For Your Reference: Volume and Surface Area Formulas

Geometric Object	Diagram	Volume	Surface Area
Rectangular Prism		$V = w \cdot h \cdot L$	$A = 2 \cdot w \cdot h + 2 \cdot w \cdot L + 2 \cdot h \cdot L$
Cylinder		$V = \pi \cdot r^2 \cdot h$	If the cylinder does not have ends, then the surface area is: $A = 2 \cdot \pi \cdot r \cdot h$ Each circular end (if present) will contribute an area of $\pi \cdot r^2$.
Cone		$V = \frac{1}{3} \cdot \pi \cdot r^2 \cdot h$	If the cones does not have a circular base, then the surface area is: $A = \pi \cdot r \cdot \sqrt{r^2 + h^2}$ The circular base (if present) contributes an area of $\pi \cdot r^2$.
Sphere		$V = \frac{4}{3} \cdot \pi \cdot r^3$	$A = 4 \cdot \pi \cdot r^2$

Two other formulas that are often very useful are the **area** ($A = \pi \cdot r^2$) and **circumference** ($C = 2 \cdot \pi \cdot r$) of a circle that has radius r.