SOLUTIONS

Recitation Handout 7: Finding the Limits of L'Hopital's Rule

The Marquis de l'Hopital (1661-1704).

L'Hopital's rule is a powerful tool that can help you to calculate difficult limits with much greater ease. The rule uses a relationship between limits of quotients and limits of quotients of derivatives to simplify limit calculations under many circumstances.

L'Hopital's rule is named for a French nobleman, the Marquis de l'Hôpital, who published the first calculus textbook in 1696. This rule appeared (for the first time in print) in that textbook. However, many historians believe that the Marquis de l'Hôpital) did not actually discover the rule. The Marquis de l'Hôpital had hired a brilliant Swiss mathematician named Johann Bernoulli to teach him calculus. There is historical evidence in some of Bernoulli's earlier writings to suggest that he knew the rule some years before teaching the Marquis calculus, although he never published the discovery.

The goals of this recitation are for you to:

- Learn the statement of L'Hopital's rule and how to apply it.
- Learn how to manipulate and adapt limits to put them into a format that allows L'Hopital's rule to be applied.
- Recognize that L'Hopital's rule is not a "one size fits all" tool that can be used to calculate every single limit.
- Learn how to recognize when (and when not) L'Hopital's rule applies in a limit calculation.

What is L'Hoptial's Rule?

L' Hopital's rule can be stated very concisely. The following theorem is the statement of the rule that is included in many calculus textbooks.

THEOREM: If $f(x)$ and $g(x)$ are differentiable functions and $f(a) = g(a) = 0$ then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the left exists.

SOLUTIONS

In this first part of the recitation we will apply the rule in its most straightforward form to calculate limits that are (otherwise) very difficult to compute.

1. Earlier in the semester we used a graph on a calculator to guess that:

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

Use L'Hopital's rule to demonstrate that this is the case.

$\lim_{\theta \to 0} \sin(\theta) = 0$ and $\lim_{\theta \to 0} \theta = 0$ so L'Hôpital's rule can be used.

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = \lim_{\theta \to 0} \frac{\cos(\theta)}{1} = 1.$$

When you use L'Hopital's rule, you may find that $f'(a) = 0$ and $g'(a) = 0$. If this is the case, you can take derivatives a second time to calculate the limit.

2. Calculate the value of the limit:

$$\lim_{t \to 0} \frac{e^t - 1 - t}{t^2}.$$

$\lim_{t \to 0} e^t - 1 - t = 0$ and $\lim_{t \to 0} t^2 = 0$ so L'Hôpital's rule can be used.

$$\lim_{t \to 0} \frac{e^t - 1 - t}{t^2} = \lim_{t \to 0} \frac{e^t - 1}{2t} = \lim_{t \to 0} \frac{e^t}{2}$$

$$= \frac{1}{2}.$$

So long as you always get zero on **both** top and bottom when you take the limit, you can usually keep taking the derivative as many times as you need to in order to find the limit.

SOLUTIONS

When preparing to apply L'Hopital's rule, it is important to remember that you may only apply it in circumstances where the numerator approaches zero and the denominator approaches zero at the same point. Sometimes it is possible to bring this about by algebraically simplifying or manipulating the limit you are given.

3. Use L'Hopital's rule to find the value of the limit given below. It may be helpful to simplify the algebraic expression before applying L'Hopital's rule.

$$\lim_{t \to 0^+} \left(\frac{1}{t} - \frac{1}{e^t - 1} \right).$$

Note that $\quad \dfrac{1}{t} - \dfrac{1}{e^t - 1} = \dfrac{e^t - 1 - t}{t \cdot (e^t - 1)}.$

Now $\quad \lim_{t \to 0} e^t - 1 - t = 0 \quad$ and $\quad \lim_{t \to 0} t \cdot (e^t - t) = 0$

so L'Hôpital's rule can be used.

$$\lim_{t \to 0^+} \frac{e^t - 1 - t}{t \cdot e^t - t} = \lim_{t \to 0^+} \frac{e^t - 1}{e^t + te^t - 1} \quad \frac{0}{0} \text{ form}$$

$$= \lim_{t \to 0^+} \frac{e^t}{e^t + e^t + te^t}$$

$$= \frac{1}{2}$$

SOLUTIONS

Limits Involving Infinity

Earlier in the semester, we spent quite a bit of time working with limits that involve infinity (including limits where $x \to \infty$ and limits in which some part of the expression we considered became infinite at some finite x-value).

L'Hopital's rule can also be employed to simplify the calculation of limits involving infinity, although it is not directly applicable for limits where $x \to \infty$. The version of L'Hopital's rule for limits involving infinity is summarized in the following theorem.

THEOREM: If $f(x)$ and $g(x)$ are differentiable functions, a is a finite number and

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty,$$

then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

Use this version of L'Hopital's rule to calculate the limits shown below. The following differentiation formulas might be helpful:

$$\frac{d}{dx}(\tan(x)) = \sec^2(x) \qquad \frac{d}{dx}(\sec(x)) = \sec(x) \cdot \tan(x).$$

4. $$\lim_{x \to \frac{\pi}{2}} \frac{\sec(x)}{\tan(x)} = \lim_{x \to \frac{\pi}{2}} \frac{\sec(x) \cdot \tan(x)}{\sec(x) \cdot \sec(x)}$$

$$= \lim_{x \to \pi/2} \frac{\sin(x)}{\cos(x)} \cdot \frac{\cos(x)}{1} = 1$$

5. $$\lim_{x \to \frac{\pi}{2}^-} \frac{\tan(x)}{\ln(\cos(x))} = \lim_{x \to \frac{\pi}{2}^-} \frac{\sec^2(x)}{-\sec(x) \cdot \sin(x)} = +\infty.$$

SOLUTIONS

Limits at Infinity

The two versions of L'Hopital's rule that we have used up to this point both contain limits expressed "as $x \to a$." In each example we have considered, a has represented some finite value (e.g. $a = \pi/2$ in Questions 4 and 5). The symbol "∞" can also be used in place of a so that L'Hopital's rule can be used for limits in which $x \to \infty$.

Generally speaking, there is never a way of using L'Hopital's rule to get around the problem of having to calculate the limit of some expression as $x \to \infty$. However, by carefully using L'Hopital's rule you can create a limit (as $x \to \infty$) that is easier to calculate than the one you started with.

5. Calculate the value of the limit:

$$\lim_{t \to \infty}\left(\frac{1}{t} - \frac{2}{t^2}\right). \qquad \frac{1}{t} - \frac{2}{t^2} = \frac{t-2}{t^2}, t \neq 0$$

$$\lim_{t \to \infty} \frac{t-2}{t^2} = \lim_{t \to \infty} \frac{1}{2t} = 0.$$

6. Was it (or would it have been) helpful to immediately differentiate the top and the bottom of each fraction in the manner that L'Hopital's rule suggests?

No, it would not have been, since neither $1/t$ nor $2/t^2$ is an indeterminant form as $t \to \infty$.

7. Was this (or would it have been) even justified according to L'Hopital's rule? Why or why not?

No it wouldn't. $1/t$ and $2/t^2$ are not indeterminant forms as $t \to \infty$.

8. Is there any way that this limit could have been evaluated more easily than by using L'Hopital's rule?

Yes. Both $\lim_{t \to \infty} \frac{1}{t} = 0$ and $\lim_{t \to \infty} \frac{2}{t^2} = 0$ exist,

So $\lim_{t \to \infty}\left(\frac{1}{t} - \frac{2}{t^2}\right) = \lim_{t \to \infty} \frac{1}{t} - \lim_{t \to \infty} \frac{2}{t^2} = 0.$

SOLUTIONS

We have already studied techniques that could have been used to evaluate the previous limit more easily. However, the next two limits are very difficult to compute without using L'Hopital's rule. Use L'Hopital to evaluate them both.

9. Calculate the value of the limit:

$$\lim_{x \to \infty} \frac{x}{e^x}.$$

$\lim_{x \to \infty} x = +\infty$ and $\lim_{x \to \infty} e^x = +\infty$ so L'Hôpital's rule can be used.

$$\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0.$$

10. Calculate the value of the limit:

$$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}}.$$

$\lim_{x \to \infty} \ln(x) = +\infty$ and $\lim_{x \to \infty} \sqrt{x} = +\infty$ so L'Hôpital's rule can be used.

$$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x \to \infty} \frac{1}{x} \cdot \frac{1}{\frac{1}{2\sqrt{x}}}$$

$$= \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0.$$

SOLUTIONS

Rewriting Limits so that we can use L'Hopital's Rule: Indeterminant Forms

The two situation that allow us to apply L'Hopital's rule to calculate a limit are ones where the limit we are trying to compute resembles one of the following "fractions:"

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}.$$

These are, of course, not mathematically meaningful fractions. Instead they are a kind of shorthand used to refer to the situations in which we can use L'Hopital's rule. These "fractions" are never called fractions, instead they are called **indeterminant forms.**

If we can find a way to rewrite a limit so that we can recognize it as one of these indeterminant forms then we can use L'Hopital's rule to evaluate the limit.

Rewrite each of the following limits in a form that allows the use of L'Hopital's rule, and then use L'Hopital's rule to evaluate each limit.

11. $\lim\limits_{x \to 0^+} x \cdot \ln(x)$.

$$= \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} \qquad \frac{\infty}{\infty} \text{ form.}$$

$$= \lim_{x \to 0^+} \frac{1}{x} \cdot \frac{1}{\frac{-1}{x^2}}$$

$$= \lim_{x \to 0^+} -x = 0.$$

12. $\lim\limits_{x \to \infty} x^{\frac{1}{x}}$.

$$= e^{\lim\limits_{x \to \infty} \ln\left(x^{1/x}\right)}$$

$$= e^{\lim\limits_{x \to \infty} \frac{1}{x} \cdot \ln(x)}$$

$$= e^{\lim\limits_{x \to \infty} \frac{1}{x}/1}$$

$$= e^{0}$$

$$= 1.$$

SOLUTIONS

13. $\lim_{x \to 0^+} x^x = e^{\lim_{x\to 0^+} \ln(x^x)}$

$$= e^{\lim_{x\to 0^+} x \cdot \ln(x)}$$

$$= e^0$$

$$= 1.$$

14. $\lim_{x \to 0}\left(\frac{1}{x} - \frac{1}{\sin(x)}\right) = \lim_{x\to 0} \frac{\sin(x) - x}{x \cdot \sin(x)}$

$$= \lim_{x\to 0} \frac{\cos(x) - 1}{\sin(x) + x\cdot\cos(x)}$$

$$= \lim_{x\to 0} \frac{-\sin(x)}{\cos(x) + \cos(x) - x\cdot\sin(x)}$$

$$= \frac{0}{2}$$

$$= 0.$$

SOLUTIONS

Recognizing when L'Hopital's Rule Does Not Apply

It is easy to get trigger-happy and start trying to use L'Hopital's rule all the time. L'Hopital's rule is not always applicable. For example, none of the following limits can be evaluated using L'Hopital's rule.

15. Explain why L'Hopital's rule cannot be used to evaluate each of the following limits.

(a) $\lim_{x \to \infty} \frac{e^{-x}}{\sin(x)}$.

The limit $\lim_{x \to \infty} \sin(x)$ does not exist, so $\frac{e^{-x}}{\sin(x)}$ does not form an indeterminant form as $x \to \infty$.

(b) $\lim_{x \to 1} \frac{x}{x-1}$

The limit $\lim_{x \to 1} x \neq 0$ so the rational function $\frac{x}{x-1}$ does not form an indeterminant form as $x \to 1$.

(c) $\lim_{x \to 1} \frac{\sin(2x)}{x}$

Again, this is not an indeterminant form as $\lim_{x \to 1} \sin(2x) = \sin(2)$ and $\lim_{x \to 1} x = 1$.

SOLUTIONS

A Very Famous Application of L'Hopital's Rule

If you are already very familiar with L'Hopital's rule and its applications, try the following problem. It is a difficult but historically and practically (a lot of mathematical finance is based on this limit, believe it or not) important.

16. Calculate the value of the limit:

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x.$$

HINT: Take the natural logarithm first, then take the limit of what you get as $x \to \infty$. When you have found the limit, make it the exponent of the special number $e \approx 2.718$.

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to \infty} e^{\ln\left(\left(1 + 1/x\right)^x\right)}$$

$$= \lim_{x \to \infty} e^{x \cdot \ln\left(1 + \frac{1}{x}\right)}$$

$$= \lim_{x \to \infty} e^{\frac{\ln(1 + 1/x)}{1/x}}$$

$$= e^{\lim_{x \to \infty} \frac{\ln(1 + 1/x)}{1/x}}$$

$$= e^{\lim_{x \to \infty} \frac{1}{1 + 1/x} \cdot \left(\frac{-1}{x^2}\right) / \frac{-1}{x^2}}$$

$$= e^{\lim_{x \to \infty} \frac{1}{1 + 1/x}}$$

$$= e^1$$

$$= e$$