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Recitation Handout 4: Introduction to Related Rates

The specific learning goals of this activity are for you to:

- Learn how to use trigonometry formulas to work out solutions to ballistics problems.
- Learn how to relate one rate of change (i.e. a derivative) to another using the Chain Rule.
- Learn how to solve realistic ballistics problems (such as those that the designers of the proposed Missile Defense Shield must solve) using a combination of trigonometry and calculus.

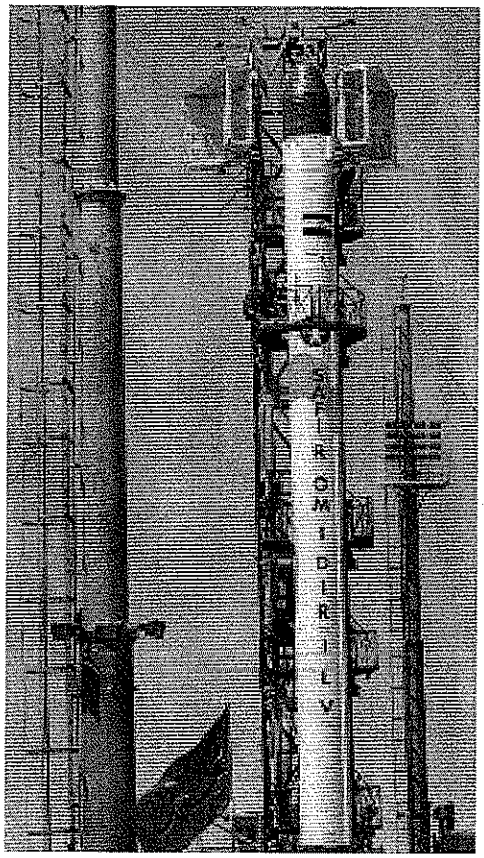

Figure 1: Iranian Safir-2 rocket being readied for launch in January, 2009.

On Saturday, February 2, 2009 Iran launched their first domestically made satellite into orbit using an Iranian rocket called the Safir-2 (see Figure 1[1]; "Safir" is the Farsi word for "Ambassador"). The satellite is a telecommunications satellite called "Omid" (Farsi for "Hope") and is the culmination of at least 10 years of Iranian efforts to develop a viable space program.

Reports of the successful launch immediately triggered warnings among Western powers, made uneasy by Iran's development of missile technology capable of putting a satellite in orbit. State Department spokesman Robert A. Wood commented, "Iran's development of a space-launch vehicle capable of putting a satellite into orbit establishes the technical basis from which Iran could develop long-range ballistic missile systems."[2]

The prospect of Iran developing nuclear weapons and long-range ballistic missiles capable of delivering those missiles to targets has raised the specter of the United State's national missile defense.

During President Clinton's and President George W. Bush's administrations, this program was fought over extensively. The current affairs program *Frontline* produced a documentary program[3] early in President Bush's first term that explored the issues surrounding a missile defense system, and you can find more information about it on-line at:

http://www.pbs.org/wgbh/pages/frontline/shows/missile

[1] Image source: http://www.space.com/

[2] Tremoglie, M. P. "Iran satellite causes fear." *The Bulletin*, February 5, 2009. Accessed on-line from: http://thebulletin.us/articles/2009/02/05/top_stories/doc498a6c49919ae78180947l.txt

[3] The program was called "Missile Wars" and aired on October 10, 2002.

Much of the fighting over the missile shield was partisan in nature and occurred in the very unusual political climate following the 9/11 attacks. However, from the debate, the opinions of some non-partisan groups appeared that criticized the missile shield concept on technical grounds. Despite the fact that many engineers openly doubted the feasibility of the missile defense system, the Department of Defense has spent more than $30 billion on it, and inspite of the fact that the system has yet to pass a realistic test, began deploying it (at the time of writing, about 20 interceptor missiles have been deployed at Ft. Greely in Alaska and Vandenburg Air Force Base in California[4]). In this recitation you will use calculus to investigate some of the technical specifications that the designers of the current generation of missile defenses.

1. Hitting a Moving Target

In this part of the recitation we will work out a very simple case of the problem that the designers of the Missile Shield have been grappling with – hitting a moving target with a projectile when both target and projectile have constant speeds and the target is moving in a straight line.

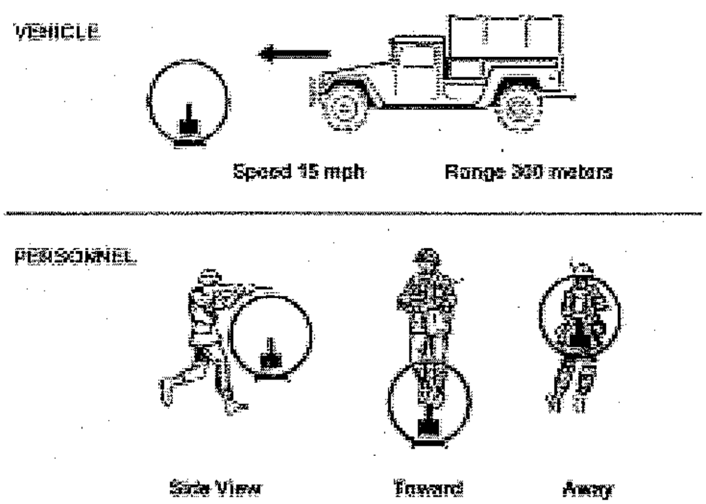

Figure 2: Educational guidelines for aspiring machine gunners, teaching them how to aim at moving targets. Note that when the soldier is running to the side, the gunner is taught to aim slightly in front of where the soldier's position.

Hitting a moving target with a projectile weapon is not an easy task, as you somehow need to take the movement of the target into account when you aim. Instead of firing directly at a moving target, you need to fire slightly ahead of the target. Figure 2 shows a training diagram given to soldiers who are learning to fire machine guns[5]. In the diagram on the lower left, the machine gunner is taught not to fire directly at the enemy soldier, but to fire slightly ahead of the soldier. This is because in the time it takes for the machine gun bullets to reach the enemy, he will have moved on a little bit. If the machine gunner fires directly at the running enemy, his rounds will probably miss.

(a) The diagram on the next page shows the situation for an anti-tank gunner attempting to hit a moving tank. Assume that the tank moves with a velocity of v_1 and the projectile from the anti-tank gun will move with a velocity of v_2. Calculate θ, the angle that the gunner should aim ahead of the tank in order to hit the tank if the gunner fires when the tank is directly in front of him or her. The only variables in your formula (besides θ) should be v_1 and v_2 and it is safe to assume that $v_2 > v_1$.

[4] Source: http://www.globalsecurity.org/space/systems/bmds.htm

[5] Source: http://www.globalsecurity.org/

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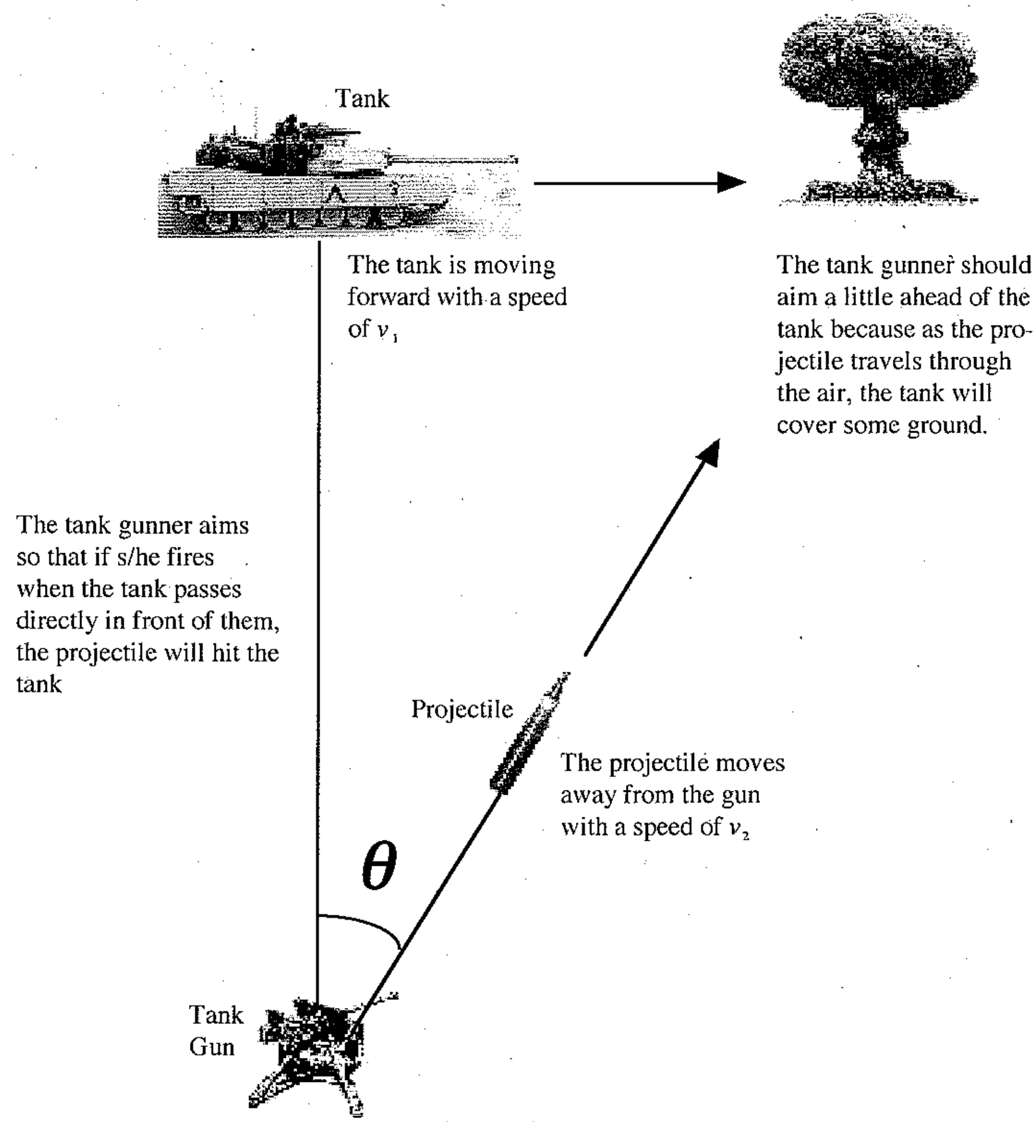
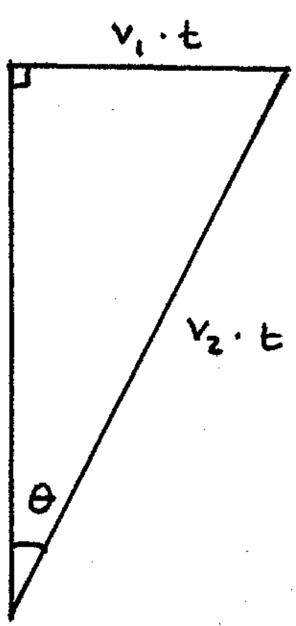


Let t = time it takes for the projectile to reach the tank.



$$\frac{v_1 \cdot t}{v_2 \cdot t} = \frac{\text{opposite}}{\text{hypoteneuse}} = \sin(\theta)$$

$$\theta = \sin^{-1}\left(\frac{v_1}{v_2}\right)$$

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(b) Soon, you will use your calculations to try to hit a moving model tank with a toy gun. As accurately as you can, measure the velocity of the tank and the velocity of the projectile for the anti-tank gun. Record your results in the table (below).

Object	Velocity (feet per second)
Tank	Value measured on 2/10/09
Anti-tank projectile	Value measured on 2/10/09.

(c) Using the velocities that you have measured, calculate the angle, θ, that the tank gunner should rotate in order to hit the moving tank. Usually, when we make trigonometric calculations in Math 120 we will want to express the angle in radians. On this one occasion, it is probably easiest to work with an answer in degrees, so make sure that convert the number from your calculator to the degree measure of angle.

Let v_1 be the value for the tank and v_2 be the value for the projectile. You plug the two values you have into:

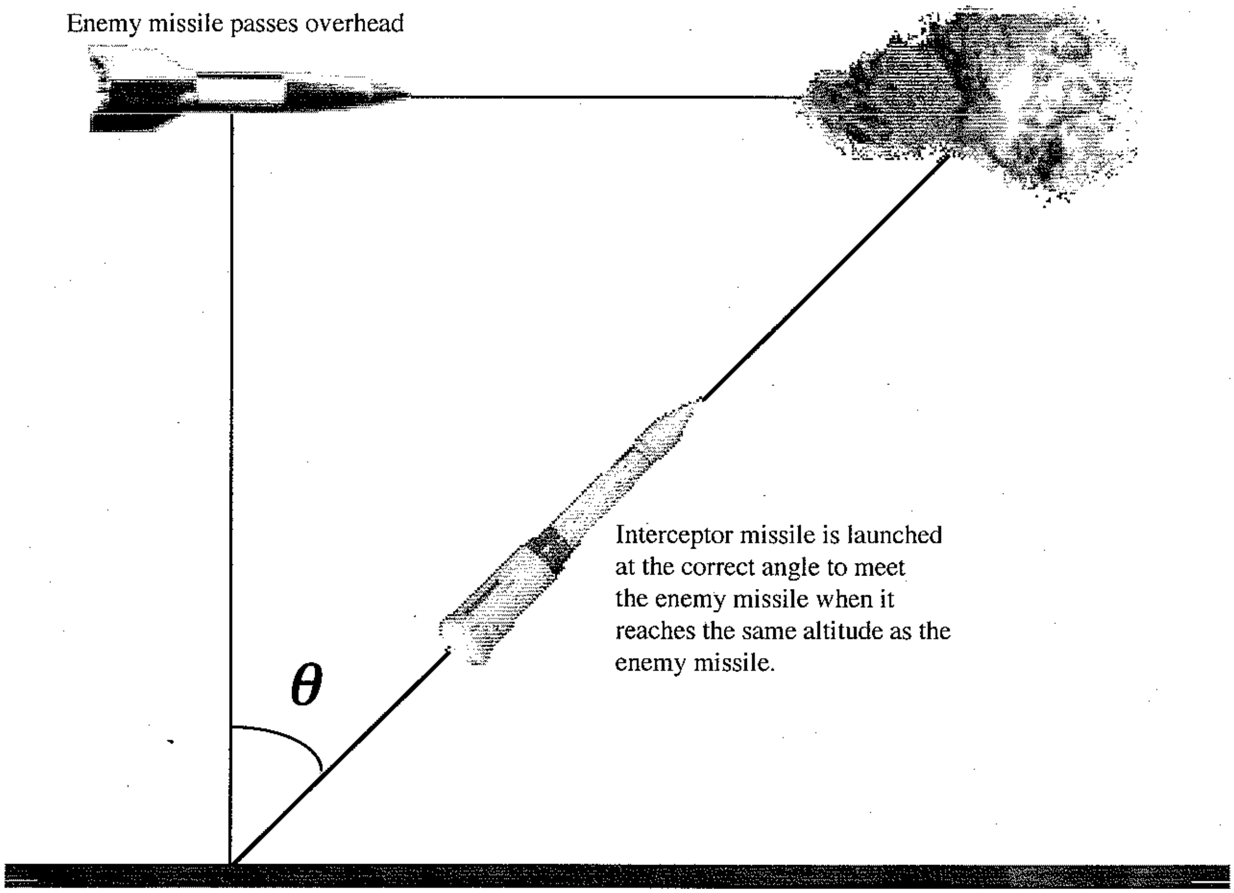
$$\theta = \sin^{-1}\left(\frac{v_1}{v_2}\right).$$

The angle will be quite small because the projectile is quite a bit faster than the tank.

(d) When you have calculated your value of θ, check your work by trying to shoot the model tank with the toy gun.

2. Keeping Your Sights on a Moving Target

The calculation that you carried out in Part 1 of this recitation is a specific case for how a missile defense system could operate. As illustrated in the diagram below, an interceptor missile on the ground could be launched at an angle θ as the enemy missile passes directly overhead. So long as the enemy missile flies level and in a straight line, the interceptor missile will be able to hit it. The angle, θ, would be calculated using the speed of the enemy missile, v_1, and the speed of the interceptor missile, v_2, using the formula that you found earlier.



A real missile defense system would probably not be designed like this. For example, the defeat this kind of system, all that an enemy would have to do is avoid flying directly over the interceptor missile launch stations. Instead, a more likely system would track incoming enemy missiles with something like radar.

(e) The diagram on the next page shows a possible set-up for a radar station that would track incoming missiles. In order to maintain a radar lock on an incoming missile, the radar dish would have to tilt (increasing the angle φ) as the missile came closer. Find a formula for the rate of change of the angle φ if the radar station is tracking an Iraqi Al-Hussein missile (the missile most commonly used during "Scud Attacks" during the 1991 Gulf War, which flies at a speed of approximately 3800 miles per hour, with an altitude of approximately 90 miles.

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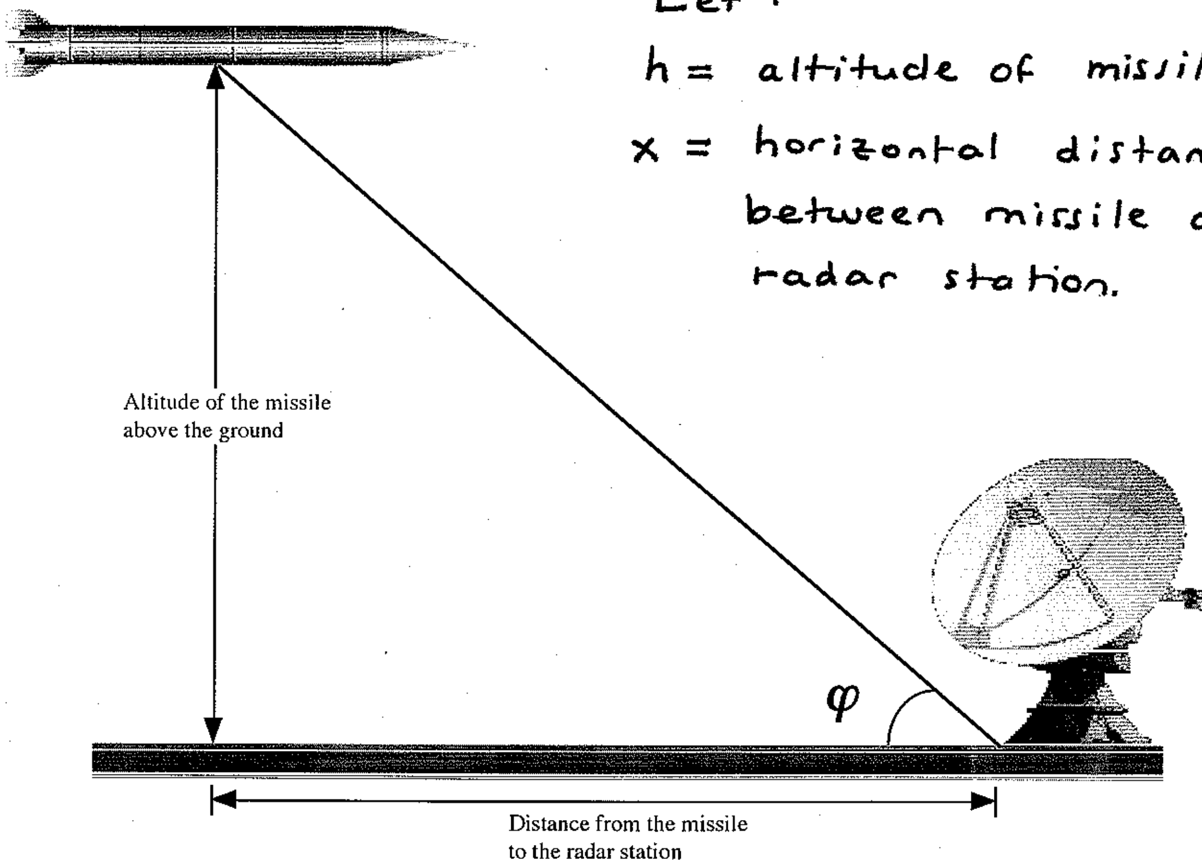
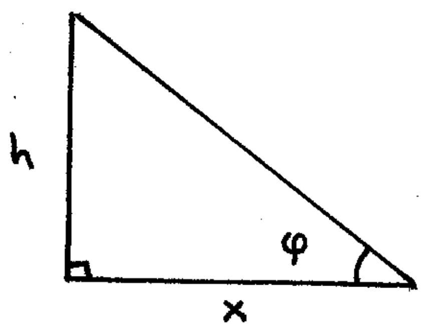
Let:

h = altitude of missile

x = horizontal distance between missile and radar station.



HINT: The derivative of $\tan(\varphi)$ is: $\dfrac{1}{\cos^2(\varphi)}$.



Using trigonometry:

$$\tan(\varphi) = \frac{h}{x}.$$

We want $\dfrac{d\varphi}{dt}$. Taking the derivative of both sides with respect to t gives:

$$\frac{1}{\cos^2(\varphi)} \cdot \frac{d\varphi}{dt} = \frac{x \cdot \frac{dh}{dt} - h \cdot \frac{dx}{dt}}{x^2} = \frac{-h}{x^2} \cdot \frac{dx}{dt}$$

if the missile remains at a constant altitude.

$$\frac{d\varphi}{dt} = \frac{-h \cdot \cos^2(\varphi)}{x^2} \cdot \frac{dx}{dt}$$

The rate of change of φ with respect to t is:

$$\frac{d\varphi}{dt} = \frac{-h \cdot \cos^2(\varphi)}{x^2} \cdot \frac{dx}{dt} = \frac{-h}{h^2 + x^2} \cdot \frac{dx}{dt}$$

as $\cos(\varphi) = \dfrac{x}{\sqrt{h^2 + x^2}}$.

(f) The Al-Hussein missile has a range of about 190 miles. Suppose that the radar dish detected this missile when it was 190 miles away. At the instant of detection, how quickly will the radar dish have to swivel in order to stay locked on to the missile[6]?

We have: $h = 90$ $x = 190$ $\frac{dx}{dt} = -3800$ so:

$$\frac{d\varphi}{dt} = \frac{-90}{(90)^2 + (190)^2} \cdot (-3800) \simeq 7.738 \text{ radians/hour.}$$

This is equivalent to about 0.002 radias per second which is easy to accomplish with existing technology.

[6] For comparison, the motors used to move mobile radar dishes are able to produce title rates of about 1.6 radians per second. Source: Friedman, N. 1997. *The Naval Institute Guide to World Naval Weapon Systems, 1997-1998.* Annapolis, MD: Naval Institute Press.

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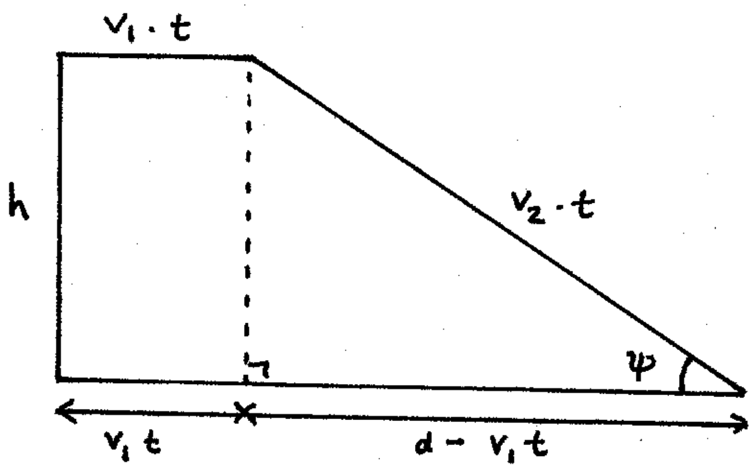
3. *Really* Keeping Your Sights on a Moving Target

In this final part of the recitation we will put the first two parts of the recitation together.

(g) Imagine that you are an engineer working for the Boeing corporation[7] and that you have been given the task of designing the launcher that will point and fire interceptor missiles at incoming enemy missiles. Your launcher must swivel and keep the interceptor missile pointed at the correct place where it must fire in order to intercept the incoming enemy missile. (Remember that this is not quite the same as pointing directly at the incoming enemy missile.) You may assume that the sensors that your launcher receives data from can tell you the speed of the enemy missile (v_1), the altitude of the enemy missile (h), and the horizontal distance to the missile (d). You will know the speed of your interceptor missile (v_2). Find a formula for the angle, ψ, that the launcher should tilt to in order to intercept an incoming missile. Your answer should be expressed in terms of h, d, v_1 and v_2.

Let t = time it takes for the interceptor missile to reach the enemy missile.



Using Pythagoras:

$$h^2 + (d - v_1 t)^2 = (v_2 t)^2$$

FOILing and rearranging:

$$(v_2^2 - v_1^2)t^2 + 2dv_1 t - (h^2 + d^2) = 0.$$

Using the quadratic formula to solve for t (and ignoring $t < 0$) gives:

$$t = \frac{-2dv_1 + \sqrt{4d^2v_1^2 + 4(v_2^2 - v_1^2)(h^2 + d^2)}}{2(v_2^2 - v_1^2)}$$

Now, $\sin(\psi) = \dfrac{h}{v_2 t}$ so:

$$\psi = \sin^{-1}\left(\frac{h}{v_2 \cdot \left(\dfrac{-2dv_1 + \sqrt{4d^2v_1^2 + 4(v_2^2 - v_1^2)(h^2 + d^2)}}{2(v_2^2 - v_1^2)}\right)}\right)$$

[7] Boeing currently has an exclusive contract with the Department of Defense to develop and manufacture equipment, systems and missiles for the Ground-based Midcourse Defense System, which is the part of the missile shield that is supposed to function\ the most like the situations we have investigated so far. You can see what Boeing has to say about their program at: http://www.boeing.com/defense-space/space/gmd/index.html