Recitation Handout 2: Solutions

1. The feature of the **graph** that suggests that the function $f(x) = 10 - 10 \cdot \left(\frac{5}{8}\right)^x$ has a limit of 10 as $x \to \infty$ is the **horizontal asymptote** at a height of 10. The feature of the **table** of values that suggests that the function $f(x) = 10 - 10 \cdot \left(\frac{5}{8}\right)^x$ has a limit of 10 as $x \to \infty$ is the fact that when x becomes large, the values of $f(x)$ recorded in the table are always approximately equal to 10.

Note that these observations are merely suggestive – neither is conclusive proof that the function $f(x) = 10 - 10 \cdot \left(\frac{5}{8}\right)^x$ actually does have a limit of 10 as $x \to \infty$. The reason for this is that some functions (like John Travolta's cubic polynomial, for example) have a plateau where the function values temporarily level off, but later start to rise again. Although it is unlikely, this could also be the case for the function $f(x)$ considered here.

The observations are suggestive because they were made over a limited range of x-values. Over the x-values examined, it certainly seems to be the case that the graph of $y = f(x)$ does level off at a height of 10. However, this is based on examining the behavior of the graph between $x = -2$ and $x = 17$. Although it is very unlikely, the graph could start to do strange and unanticipated things for higher x-values.

The only way to know whether a given function actually has a limit is to "read the algebra" of the function's equation, and see what each part of the equation will do when x approaches very large positive or very large negative values.

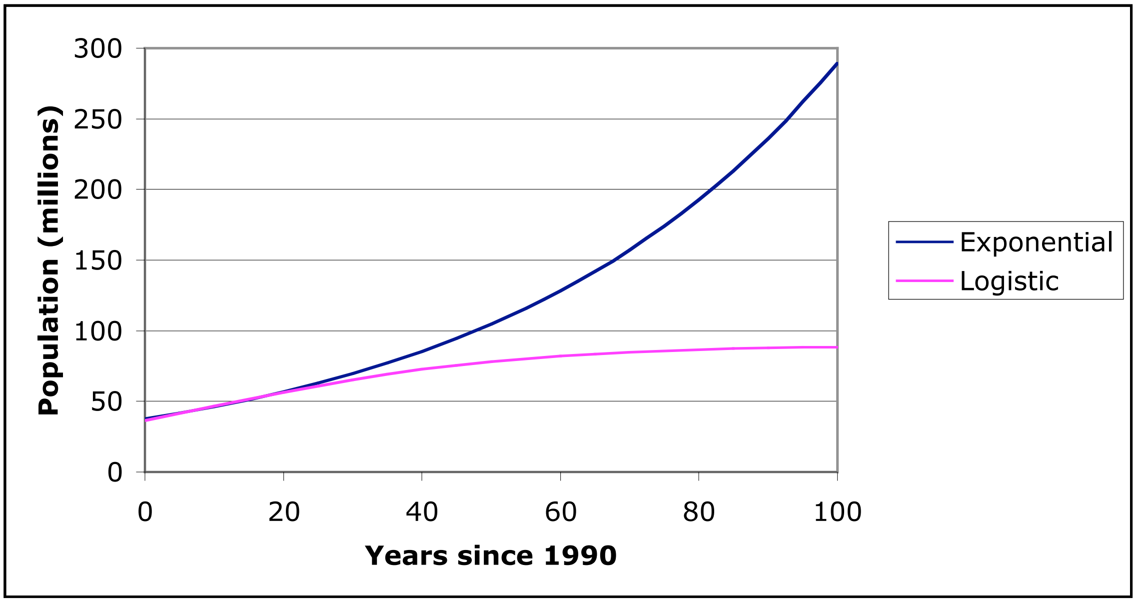
2. The graphs of the two population models,

Exponential growth: $P = (37.819) \cdot (1.02056)^x$.

Logistic growth: $P = \dfrac{89.808}{1 + (1.48) \cdot (0.9552)^x}$.

are shown below. The graph of the **exponential population function** does not show any signs that it will settle down to a steady height (as you would expect for the graph of a function that had a finite limit as $x \to \infty$). On the contrary, the graph of the **exponential population function** appears to be increasing more and more quickly as time goes by. This suggests that the limit of the exponential growth function, as $x \to \infty$, will be infinity.

On the other hand, the graph of the **logistic population function** does appear to eventually level off at a height of approximately 90. This suggests that the **logistic population function** will have a finite limit as $x \to \infty$, and that the value of this limit will be in the neighborhood of 90.



3. We will examine the algebraic structure of each function to determine what the function does as $x \to \infty$.

Exponential growth: $P = (37.819) \cdot (1.02056)^x$.

As x gets larger and large, the size of the power that 1.02056 is raised to gets larger and larger. As this growth factor, 1.02056, is greater than 1, when x grows the size of $(1.02056)^x$ also grows. There is no restriction on the size that $(1.02056)^x$ can grow to. Therefore, there is no restriction on the size that $P = (37.819) \cdot (1.02056)^x$ can attain when x is allowed to assume larger and larger values. Therefore, the limit of the (exponential version of) P is $+\infty$ when $x \to \infty$.

Logistic growth: $P = \dfrac{89.808}{1 + (1.48) \cdot (0.9552)^x}$.

As x gets larger and larger, the power that 0.9552 is raised to gets larger and larger. As this factor, 0.9552, is less than one, when x grows the size of $(0.9552)^x$ shrinks, getting closer and closer to zero.

Therefore, as x grows, the denominator of the logistic function, $1 + (1.48) \cdot (0.9552)^x$ gets closer and closer to a value of one, because $(1.48) \cdot (0.9552)^x$ gets closer and closer to a value of zero.

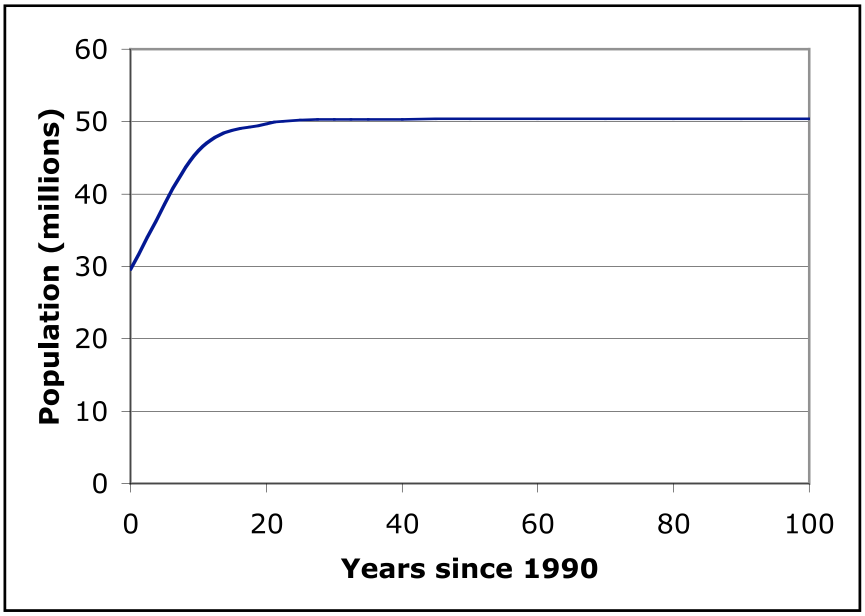
Therefore, since the value of the numerator, 89.808. does not change as x grows, the overall value of the logistic function gets closer and closer to $89.808/1 = 89.808$ in value.

Our conclusion is that the limit (as $x \to \infty$) of the logistic function

$$P = \frac{89.808}{1 + (1.48) \cdot (0.9552)^x}$$

is equal to 89.808.

4. The graph of the function that represents the size South African population in the event that the government continues with its current policies on HIV/AIDS is shown below.



From the appearance of the graph, this function definitely seems to have a finite limit as $x \to \infty$, and the value of the limit appears to be slightly higher than 50.

You can deduce the existence and precise value of this limit from the algebraic structure of the equation for the logistic function,

$$P = \frac{50.349}{1 + (0.702) \cdot (0.8196)^x}.$$

As x gets larger and larger, the power that 0.8196 is raised to gets larger and larger. As this factor, 0.8196, is less than one, when x grows the size of $(0.8196)^x$ shrinks, getting closer and closer to zero. As x grows, the denominator of the logistic function, $1 + (0.702)\cdot(0.8196)^x$ gets closer and closer to a value of one, because $(0.702)\cdot(0.8196)^x$ gets closer and closer to a value of zero. Therefore, since the value of the numerator, 50.349. does not change as x grows, the overall value of the logistic function gets closer and closer to $50.349/1 = 50.349$ in value.

5. The difference that the government policies make to South Africa's population will be given by the difference in value between the logistic function from Question 2 and the logistic function from Question 4. The values of these functions and the numerical values of the differences are shown in the table below, and the differences in the completed graphic.

Year	x	Population if HIV medications distributed (millions)	Population with current government policies (millions)	Difference (millions)
2009	19	55.453	49.555	5.898
2019	29	64.529	50.239	14.290
2059	69	84.515	50.349	34.166
Long term	$x \to \infty$	89.808	50.349	39.459