SOLUTIONS

Math 120 **Winter 2009**

Handout 6: In-Class Review for Exam 1

The topics covered by Exam 1 in the course include the following:

- Functions and their representations.
- Detecting functions from tables, formulas and graphs.
- Domains of functions (particularly when defined by formulas).
- Functions defined in pieces.
- Increasing and decreasing functions.
- Linear functions.
- Quadratic functions.
- Polynomial functions.
- Rational functions.
- Trigonometric functions (focus on sine and cosine).
- Finding formulas for functions from graphs, tables of values or verbal descriptions of situations.
- Finding composite functions.
- Domains of new functions created from old functions (e.g. composite functions).
- Limits of functions, including right and left hand limits.
- Approximating limits by making tables of values.
- Existence of limits from tables, graphs and formulas.
- Calculating limits exactly using algebra and the Squeeze Lemma.
- Limits as $x \to \pm\infty$.
- Limits where a function becomes infinite.
- Horizontal and vertical asymptotes of a function (particularly a rational function).
- Secant lines and average rates of change.
- Tangent lines and instantaneous rates of change.
- Calculating the derivative of a function at a point using the limit definition.
- Calculating a formula for the derivative of a function using the limit definition.
- Sketching the graph of the derivative when the function is defined by a graph.
- Finding the maximum and minimum points of a function by setting the derivative equal to zero.
- Calculating derivatives using the "short cut" rules.
- Calculating derivatives using the product, quotient and chain rules (using formulas, given values or graphs).
- Interpretation and units of the derivative.

1. Find the horizontal and vertical asymptotes of the rational function:

$$R(x) = \frac{x^2 - 3x + 2}{x^2 - 9}.$$

Use these values and the axes on the next page to help to sketch the graph of $y = R(x)$.

$$R(x) = \frac{(x-2)(x-1)}{(x-3)(x+3)}$$

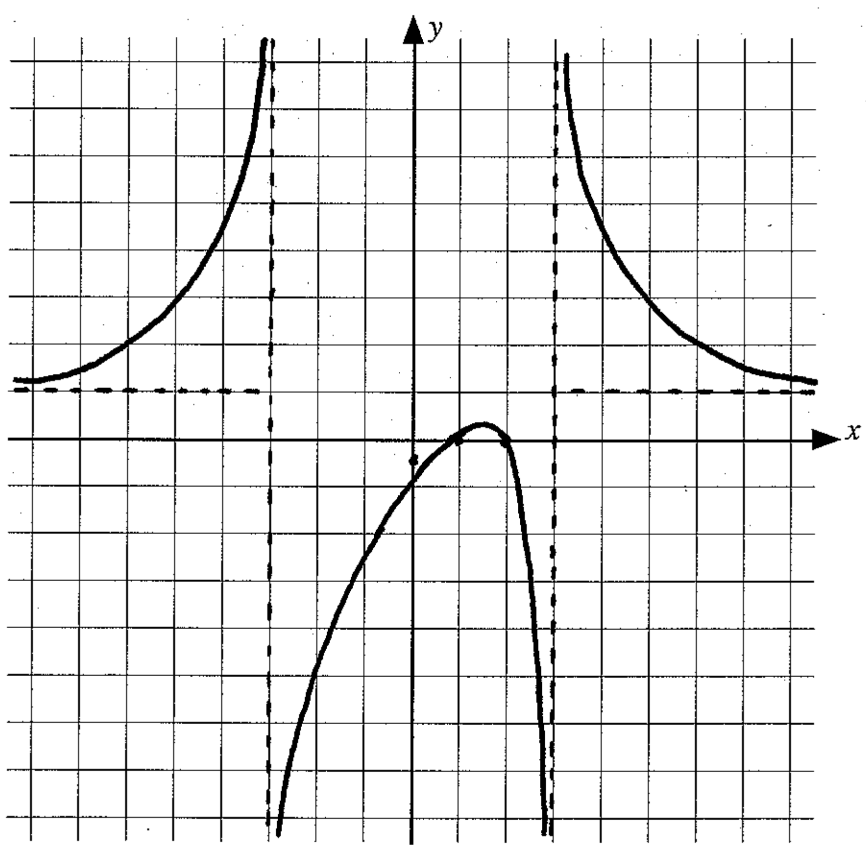
Vertical Asymptotes occur at $x = 3$ and $x = -3$.

SOLUTIONS.

Horizontal asymptotes:

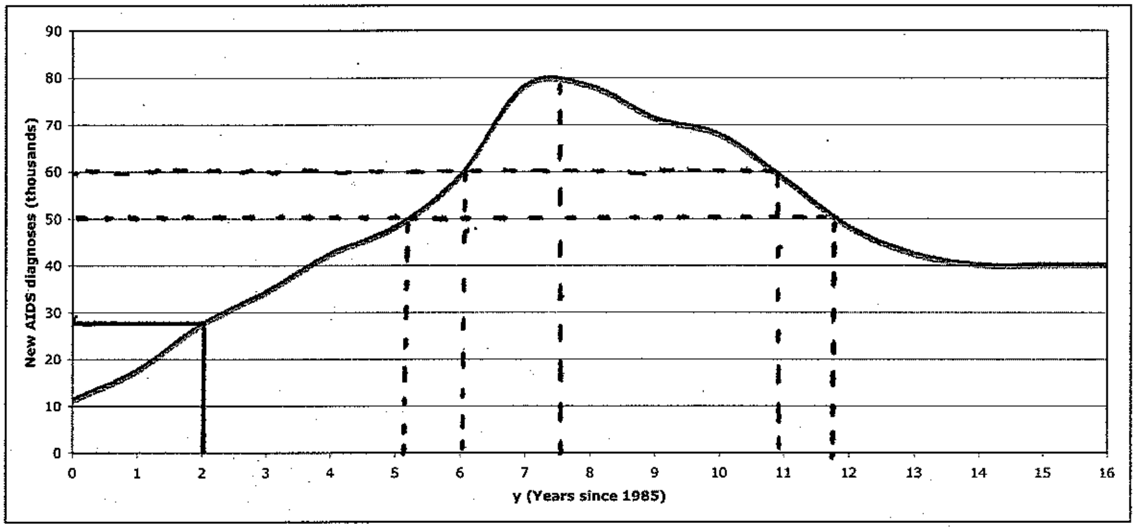
$$\lim_{x \to \infty} R(x) = +1$$

$$\lim_{x \to -\infty} R(x) = +1$$



SOLUTIONS

2. The graph shown below gives the number of new AIDS diagnoses in the United States. This graph is the graph of the function $A = f(y)$, where A is the number of new AIDS diagnoses (in thousands) and y is the number of years since 1985.



(a) Use the graph provided to estimate $f(2)$ and explain the practical meaning of this quantity in terms of the AIDS epidemic.

$f(2) \approx 27$

This means that in the year 1987, there were 27,000 new AIDS diagnoses in the United States.

(b) Approximately how many new AIDS diagnoses were made when the epidemic was at its height? In what year was the epidemic at its height?

The epidemic was at its height in 1992 or 1993. During this year there were approximately 80,000 new AIDS diagnoses in the United States.

(c) Use the graph provided above to estimate the solution(s) of the equation $f(y) = 50$ and explain the practical meaning of any solution(s) in terms of the AIDS epidemic.

The solutions of $f(y) = 50$ are $y \approx 5$ and $y \approx 12$. This means that in both the years 1990 and 1997 there were 50,000 (approx.) new AIDS diagnoses in the United States.

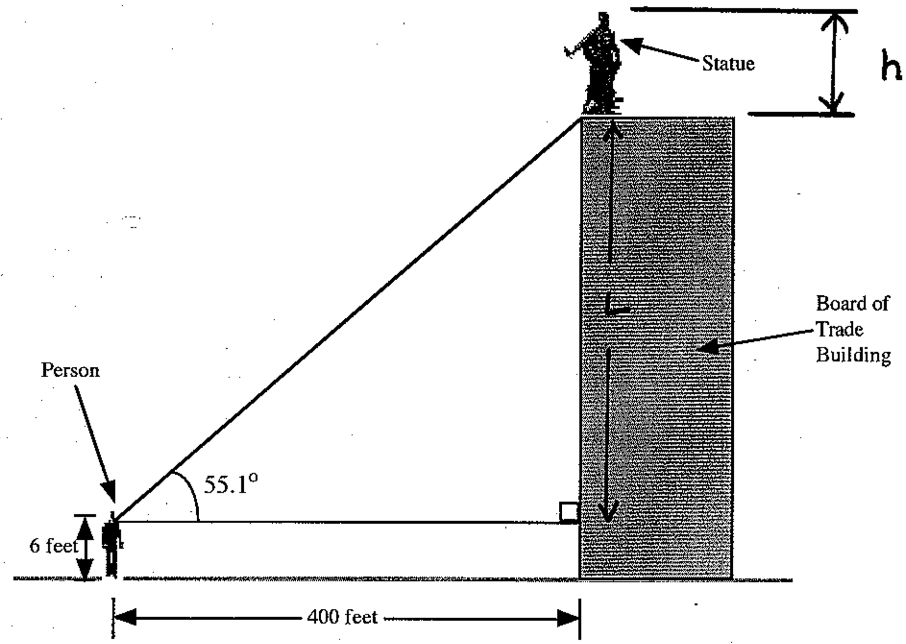
(d) Use the graph provided above to estimate the solution of the inequality: $f(y) \geq 60$.

The solutions of the inequality are:

$$6 \leq y \leq 11.$$

SOLUTIONS

3. The Board of Trade building in Chicago, IL, has a statue on top of it. The statue depicts Ceres, the Roman goddess of harvests. A person is standing on the ground 400 feet from the building looking up at the statue. The person is six (6) feet tall. First, the person looks at the feet of the statue and notes that their line of sight makes an angle of 55.1°. Next, the person looks at the head of the statue and notes that their line of sight makes an angle of 56.5°. How tall is the statue of Ceres? Show your work and explain your reasoning.



$$\frac{L}{400} = \tan(55.1°)$$

$$\frac{L + h}{400} = \tan(56.5°)$$

$$h = 400 \cdot \tan(56.5°) - 400 \cdot \tan(55.1°)$$

$$= 30.9475 \text{ feet.}$$

SOLUTIONS

4. In each of the following cases, find the value of the constant k so that the given limit exists.

(a) $\lim_{x \to 4} \frac{x^2 - k^2}{x-4}$

Want $x^2 - k^2 = (x+k)(x-k)$ to have a factor that will cancel the $x-4$ in the denominator.

$k = 4$ or $k = -4$.

(b) $\lim_{x \to 1} \frac{x^2 - kx + 4}{x-1}$

Want $x^2 - kx + 4$ to factor so that it has a factor of $x-1$ to cancel the $x-1$ in the denominator.

$k = 5$.

(c) $\lim_{x \to -2} \frac{x^2 + 4x + k}{x+2}$

Want $x^2 + 4x + k$ to have a factor of $x+2$ to cancel the $x+2$ in the denominator.

$k = 4$.

(d) $\lim_{x \to \infty} \frac{x^2 + 3x + 5}{4x + 1 + x^k}$

Want a power of at least x^2 in the denominator to compensate for the x^2 in the numerator.

$k \geq 2$.

(e) $\lim_{x \to \infty} \frac{x^3 - 6}{x^k + 3}$

Want a power of at least x^3 in the denominator to compensate for the x^3 in the numerator.

$k \geq 3$.

SOLUTIONS

5. For each of the functions listed below, find a formula for $f'(x)$.

(a) $f(x) = x^2 - 3x + 7$

Difference Quotient:

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - 3(x+h) + 7 - (x^2 - 3x + 7)}{h}$$

$$= \frac{x^2 + 2xh + h^2 - 3x - 3h + 7 - x^2 + 3x - 7}{h}$$

$$= \frac{2xh + h^2 - 3h}{h}$$

$$= 2x + h - 3 \quad , \quad h \neq 0.$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} 2x + h - 3$$

$$= 2x - 3.$$

(b) $f(x) = \dfrac{1}{2x+1}$

Difference Quotient :

$$\frac{f(x+h) - f(x)}{h} = \frac{\dfrac{1}{2(x+h)+1} - \dfrac{1}{2x+1}}{h}$$

$$= \frac{\dfrac{2x+1 - (2x+2h+1)}{((2x+h)+1)\cdot(2x+1)}}{h}$$

$$= \frac{-2h}{h\cdot(2x+2h+1)\cdot(2x+1)}$$

$$= \frac{-2}{(2x+2h+1)\cdot(2x+1)}, \quad h \neq 0$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{-2}{(2x+2h+1)\cdot(2x+1)}$$

$$= \frac{-2}{(2x+1)^2}$$

(c) $f(x) = \sqrt{x+2}$

Difference Quotient:

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h}$$

$$= \left(\frac{\sqrt{x+h+2} - \sqrt{x+2}}{h}\right)\left(\frac{\sqrt{x+h+2} + \sqrt{x+2}}{\sqrt{x+h+2} + \sqrt{x+2}}\right)$$

$$= \frac{x+h+2 - (x+2)}{h\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}$$

$$= \frac{h}{h\left(\sqrt{x+h+2} + \sqrt{x+2}\right)}$$

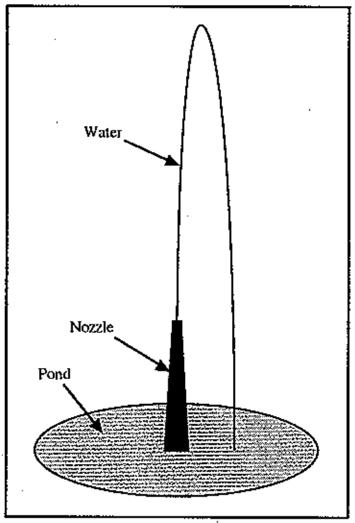
$$= \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}}, \quad h \neq 0$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}}$$

$$= \frac{1}{2\sqrt{x+2}}$$

SOLUTIONS



6. A fountain shoots water up into the air. The fountain consists of a pond and a nozzle. The water is shot from the top of the nozzle (see diagram).

The height of the water that is shot from the nozzle is given by the quadratic function:

$$H(t) = -16t^2 + 112t + 10,$$

where $H(t)$ is the height of the water above the surface of the pond in feet t seconds after it has left the top of the nozzle.

(a) How high is the top of the nozzle above the surface of the pond? Give your answer in feet.

The height of the nozzle is given by $H(0)$.

$H(0) = 10$ feet.

(b) When water is fired out of the nozzle, it rises up in the air and then falls back down into the pond. How many seconds (after it is fired out of the top of the nozzle) does it take for the water to reach the surface of the pond?

The time, t, will be the solution of the quadratic equation:

$$-16t^2 + 112t + 10 = 0$$

$$t = \frac{-112 \pm \sqrt{112^2 - 4(-16)(10)}}{(2)(-16)} = -0.088 \text{ or } 7.088.$$

Of the two solutions, $t = 7.088$ corresponds to the time that the water spends in the air before it hits the surface of the pond.

SOLUTIONS

(c) How many seconds (after it is fired out of the top of the nozzle) does it take for the water to reach its maximum height?

$$H'(t) = -32t + 112$$

To find t for maximum height, sove $H'(t) = 0$ for t.

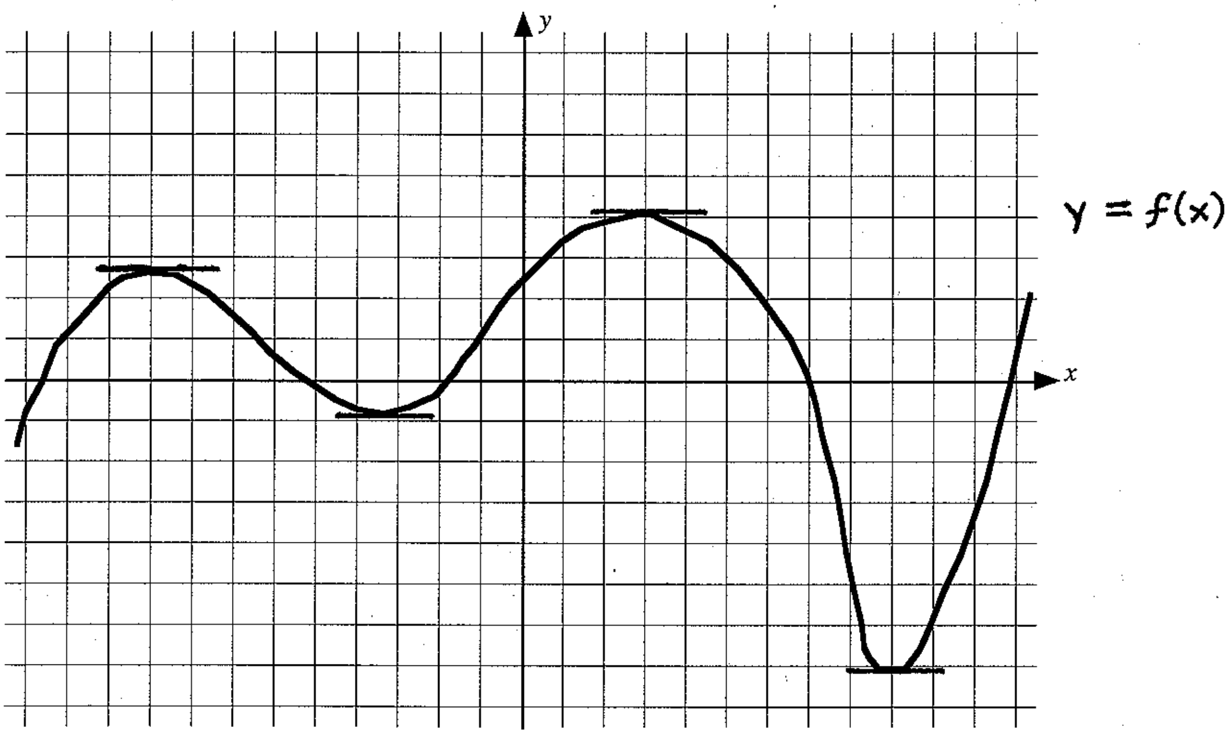
$$t = \frac{112}{32} = 3.5 \text{ seconds.}$$

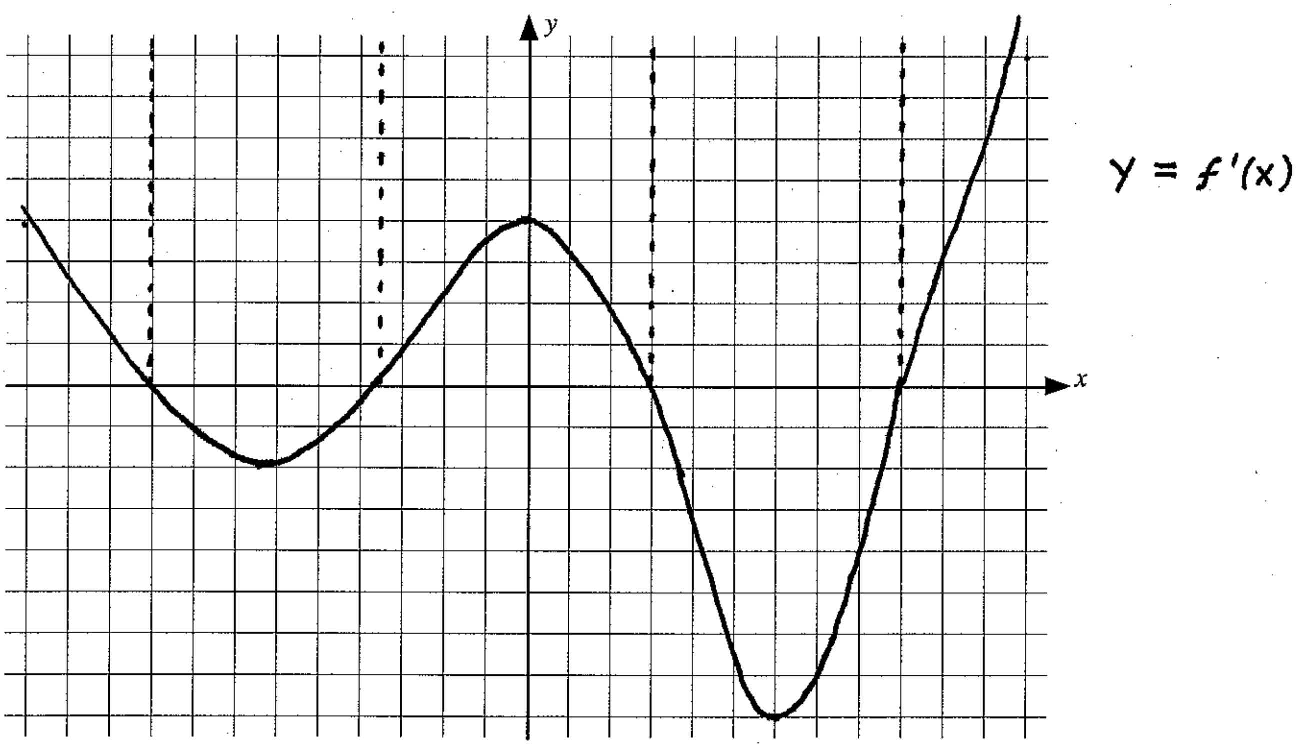
(d) What is the maximum height above the surface of the pond that the water reaches?

$$H(3.5) = 206 \text{ feet.}$$

SOLUTIONS

7. The graph of $y = f(x)$ is given below. Use the axes provided to sketch the graph of $y = f'(x)$.





SOLUTIONS

8. The size of a herd of reindeer is modeled by the function:

$$P(t) = 4000 + 400 \cdot \sin\left(\frac{\pi}{6} t\right),$$

where t is measured in months starting at April 1.

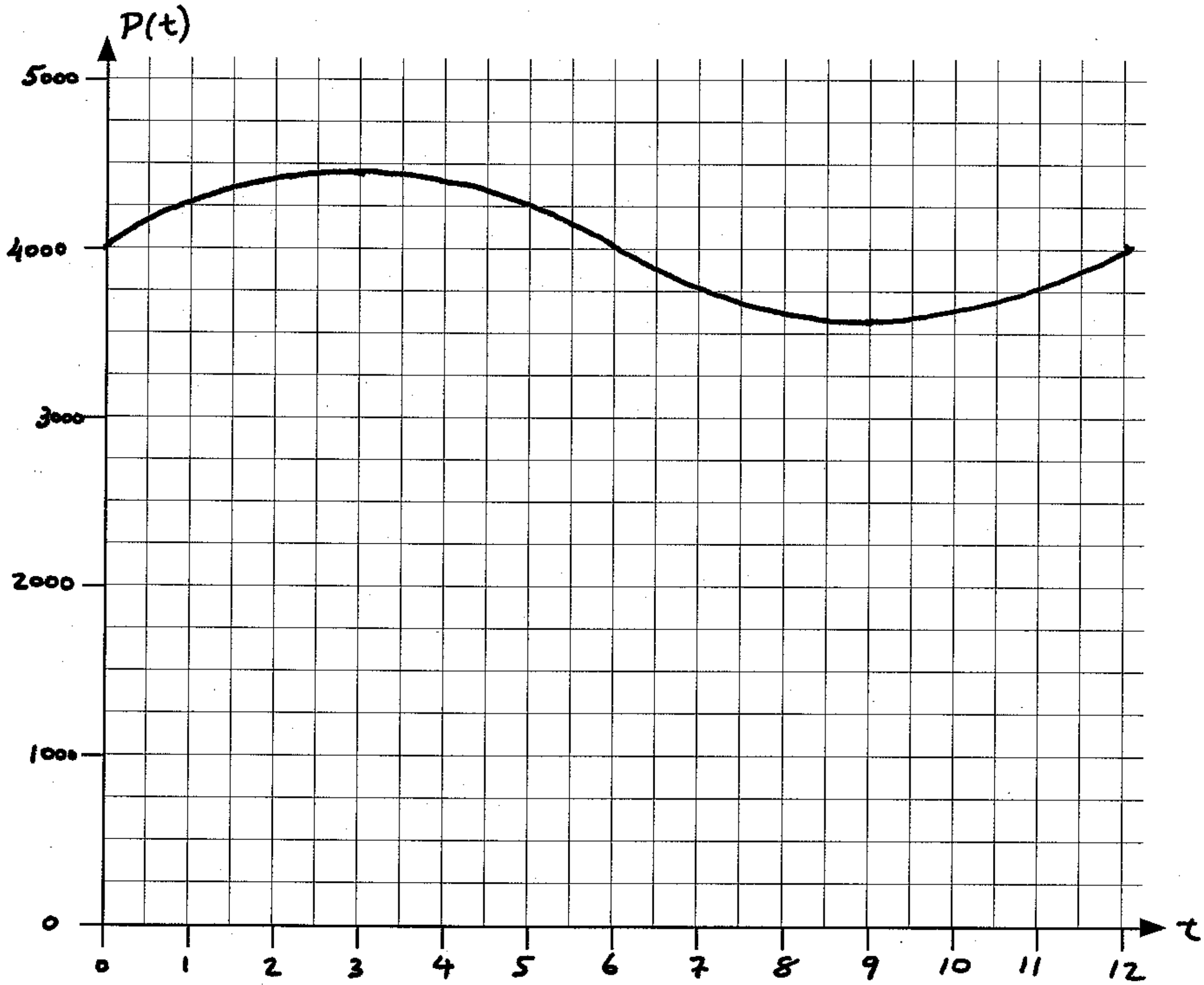
(a) What is the midline, amplitude and period for $P(t)$?

Midline: $y = 4000$

Amplitude $= 400$

Period $= 12$

(b) Use the axes provided below to sketch the graph of $P(t)$ between $t = 0$ and $t = 12$.



SOLUTIONS

(c) When (value of t and month of the year) is the herd the largest?

The herd is largest when $t = 3$.

This will correspond to the month of July.

You can calculate this by solving the equation: $P'(t) = 400 \cdot \cos\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6} = 0.$

The solutions are: $\frac{\pi}{6}t = \frac{\pi}{2}$ or $t = 3$

and: $\frac{\pi}{6}t = \frac{3\pi}{2}$ or $t = 9$.

(d) When (value of t and month of the year) is the herd the smallest?

The herd is smallest when $t = 9$.

This will correspond to the month of January.

You can tell this is the minimum by the appearance of the graph near the solution (from Part (c)) $t = 9$ of the equation $P'(t) = 400 \cdot \cos\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6} = 0.$

(e) Find the value of $P'(2)$ and give a practical interpretation of what this number means. Include units in your answer.

$$P'(2) = 400 \cdot \cos\left(\frac{\pi}{6} \cdot 2\right) \cdot \frac{\pi}{6}$$

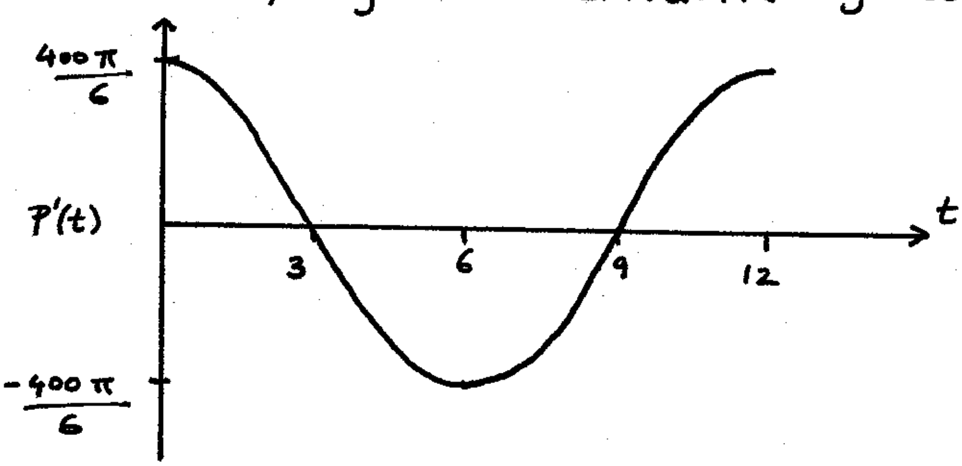
$$= \frac{100\pi}{3} \text{ deer per month.}$$

$$\doteqdot 105$$

This means that from June $(t = 2)$ to July $(t = 3)$, the size of the herd increases by approximately 105 reindeer.

(f) When is the size of the herd increasing most rapidly? When is it decreasing most rapidly?

Graphing the derivative gives something like:



The derivative is greatest at $t = 0$ and $t = 12$. This is when the size of the herd is increasing most rapidly.

The derivative is least at $t = 6$. This is when the size of the herd is decreasing most rapidly.