Handout 3: Finding a Formula for a Polynomial Using Roots and Multiplicities

A polynomial function is any function of the form:

$$y = c_0 + c_1 \cdot x + c_2 \cdot x^2 + \ldots + c_n \cdot x^n$$

where the powers of $\underline{x}$ must be positive integers. The largest power of x in the polynomial is called the **degree** of the polynomial. Most graphing calculators can only fit a very limited number of polynomial functions to data. (For example, the TI-83 can only fit polynomials up to degree 4.)

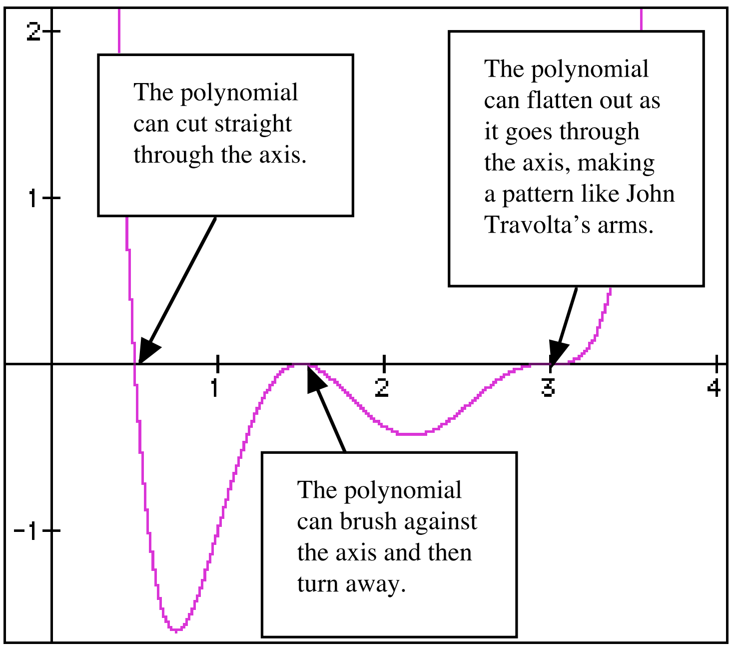

Figure 1: The shapes that a polynomial function can make as it cuts through or brushes against the *x*-axis.

Luckily, it is often possible to use the graph to find an equation for a polynomial function. The key is that you have to be able to see all of the places where the graph cuts the *x*-axis, and you have to be able to clearly see the shape that the polynomial makes as it cuts through (see Figure 1).

The basic procedure is to:

1. Locate the "x-intercepts" or zeros of the polynomial function.

2. Determine the multiplicity of each zero.

3. Write down the "factored form" of the polynomial.

4. Use a point on the graph of the polynomial to determine the constant of proportionality, *k*.

The example that follows illustrates the steps in this procedure.

Example: Finding the Equation for a Polynomial Function by Hand

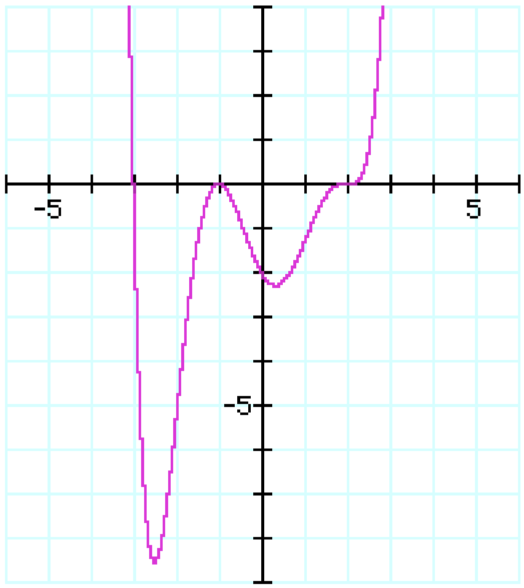

Figure 2: Example for finding the equation of a polynomial function by hand.

Step 1: Locate the zeros

Inspection of Figure 2 shows that the zeros of this polynomial are located at $x = -3$, $x = -1$ and $x = 2$.

Step 2: Determine the Multiplicity of Each Zeros

The multiplicity of the zero is determined by the appearance of graph near the zero (See Figure 3).

- If the graph looks as though it just cuts cleanly through the x-axis, then the zero has multiplicity one (see Figure 3(a)).
- If the graph looks like a quadratic and just touches the x-axis without cutting through, then the zero has multiplicity two (see Figure 3(b)).
- If the graph looks like a cubic and has an inflection point as it cuts through the x-axis, then the zero has multiplicity three (see Figure 3(c)).

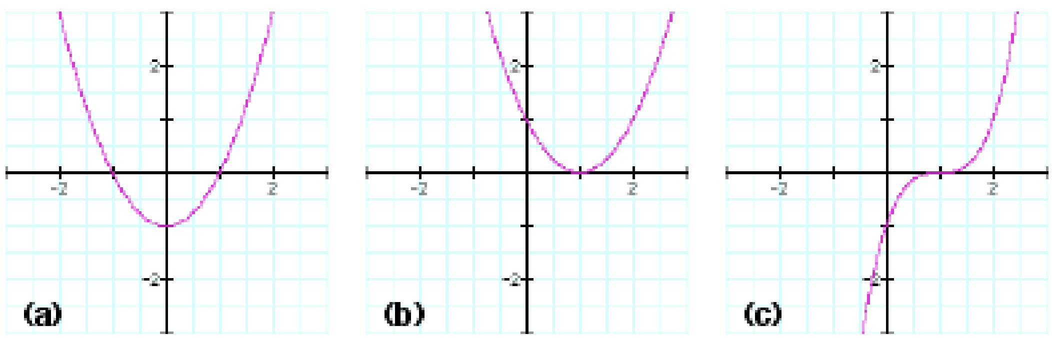

Figure 3: (a) This polynomial has two zeros, each with multiplicity 1. (b) This polynomial has one zero with multiplicity equal to 2. (c) This polynomial has one zero of multiplicity 3.

For the function graphed in Figure 2, the multiplicity of the zeros are:

Zero located at ...	Multiplicity
x = −3	1
x = −1	2
x = 2	3

Step 3: Determine the Factored Form of the Polynomial

The factored form of the polynomial is an equation of the form:

$$y = k \cdot (x - z_1)^{m_1} \cdot (x - z_2)^{m_2} \ldots (x - z_n)^{m_n},$$

where k is called the "constant of proportionality, z_1, z_2, ..., z_n are the zeros of the polynomial function, and m_1, m_2, ..., m_n are the multiplicities of the zeros. The factored form of the polynomial from Figure 2 is:

$$y = k \cdot (x + 3) \cdot (x + 1)^2 \cdot (x - 2)^3.$$

Step 4: Determine the Constant of Proportionality, *k*

The idea here is to locate the x and y coordinate of a point that is on the graph of the polynomial function, but which is not one of the zeros of the polynomial function. The x and y are substituted into the factored form, allowing k to be found.

From Figure 2, the point (0, −2) lies on the graph of the function. Substituting this into the factored form gives: $k = 1/12$. Therefore, the equation for the polynomial function whose graph is show in Figure 2 is:

$$y = \tfrac{1}{12} \cdot (x + 3) \cdot (x + 1)^2 \cdot (x - 2)^3.$$

Example

Find the formula of the polynomial function shown in Figure 4 (see over).

Solution

The zeros of the polynomial from Figure 4 are located at:

$x = -3$, and $x = +2$.

From the appearance of the graph in Figure 4 as the polynomial cuts through the x-axis, the multiplicities of the zeros are as follows:

Zero	Multiplicity
$x = -3$	1
$x = +2$	2

So, the factored form of the polynomial shown in Figure 4 is:

$$y = k \cdot (x - (-3))^1 \cdot (x - 2)^2.$$

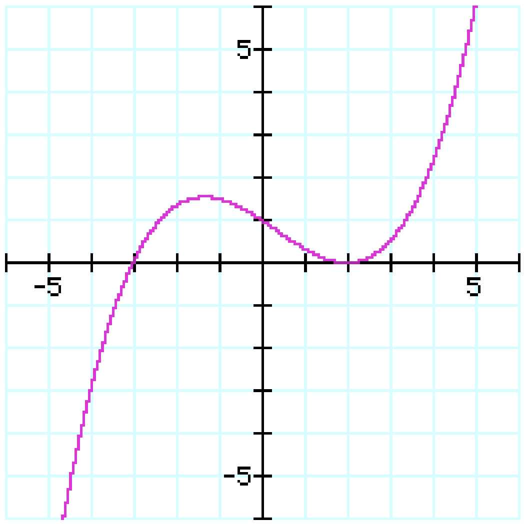

Figure 4: Find the equation of this polynomial function.

Re-writing this in a more conventional format gives the formula:

$$y = k \cdot (x+3) \cdot (x-2)^2.$$

To determine the numerical value of k we can use the fact that Figure 4 shows that the point (0, 1) lies on the graph of the polynomial function. To determine k we will substitute $x = 0$ and $y = 1$ into the factored form and solve for k. Doing this:

$$1 = k \cdot (0+3) \cdot (0-2)^2$$

$$1 = k \cdot 12$$

$$k = \frac{1}{12}.$$

With the value of k worked out, you can write down the complete formula for the polynomial function shown in Figure 4.

$$y = \tfrac{1}{12} \cdot (x+3) \cdot (x-2)^2.$$

Example

Find the formula for the polynomial function shown in Figure 5 (see below).

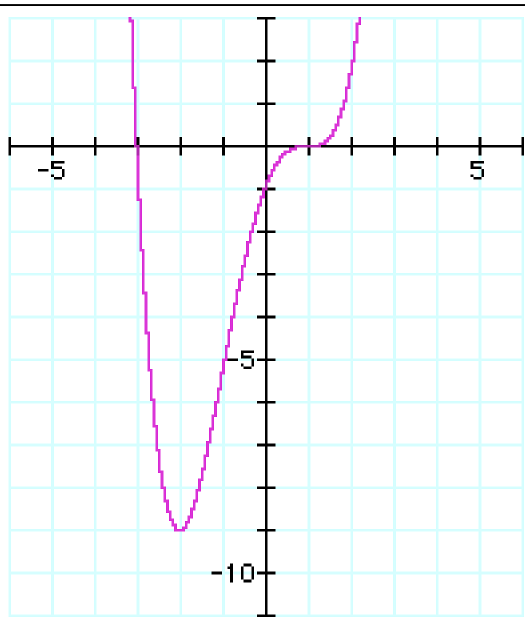

Figure 5: Find the equation of this polynomial function.

Solution

From Figure 5, the zeros of the polynomial function are:

$$x = -3 \text{ and } x = +1.$$

The multiplicities of these zeros are 1 (for $x = -3$) and 3 (for $x = +1$).

The factored form of the formula for the polynomial in Figure 5 is then:

$$y = k \cdot (x-(-3))^1 \cdot (x-1)^3.$$

Writing this formula in slightly more conventional notation gives:

$$y = k \cdot (x+3) \cdot (x-1)^3.$$

To determine the numerical value of the constant k, you can use the fact that the point (0, −1) lies on the graph.

To determine the numerical value of k you can plug $x = 0$ and $y = -1$ into the factored form, evaluate all of the quantities that you can, and then solve for k. Doing this:

$$-1 = k \cdot (0+3) \cdot (0-1)^3$$

$$-1 = k \cdot (-3)$$

$$k = \frac{1}{3}.$$

Substituting this numerical value for the k in the factored form gives the formula for the polynomial function in Figure 5:

$$y = \tfrac{1}{3} \cdot (x+3) \cdot (x-1)^3.$$