SOLUTIONS

Handout 24: Review Problems for the Final Exam

The topics covered by **cumulative final exam** may include any of the following:

- Functions and their representations.
- Detecting functions from tables, formulas and graphs.
- Domains of functions (particularly when defined by formulas).
- Functions defined in pieces.
- Increasing and decreasing functions.
- Linear functions.
- Quadratic functions.
- Polynomial functions.
- Rational functions.
- Trigonometric functions (focus on sine and cosine).
- Finding formulas for functions from graphs, tables of values or verbal descriptions of situations.
- Finding composite functions.
- Domains of new functions created from old functions (e.g. composite functions).
- Limits of functions, including right and left hand limits.
- Approximating limits by making tables of values.
- Existence of limits from tables, graphs and formulas.
- Calculating limits exactly using algebra and the Squeeze Lemma.
- Limits as $x \to \pm\infty$.
- Limits where a function becomes infinite.
- Horizontal and vertical asymptotes of a function (particularly a rational function).
- Secant lines and average rates of change.
- Tangent lines and instantaneous rates of change.
- Calculating the derivative of a function at a point using the limit definition.
- Calculating a formula for the derivative of a function using the limit definition.
- Sketching the graph of the derivative when the function is defined by a graph.
- Finding the maximum and minimum points of a function by setting the derivative equal to zero.
- Calculating derivatives using the "short cut" rules.
- Calculating derivatives using the product, quotient and chain rules (using formulas, given values or graphs).
- Interpretation and units of the derivative.
- Implicit differentiation.
- Finding formulas for tangent lines using implicit differentiation.
- Finding points on curves where the tangent line is horizontal or vertical.
- Related rates problems.
- Finding formulas for exponential functions.
- Solving equations that involve exponential functions using logarithms.
- Solving equations that involve logarithms.
- Interpreting the meaning of functions and inverses.
- The horizontal line test.
- Finding formulas for inverses of functions.
- Finding and using derivatives of inverse functions.
- Interpreting the meaning of the derivative of an inverse.
- Finding and using derivatives of functions that include exponential or logarithmic functions.
- Finding solutions to differential equations.
- Evaluating the constants in a differential equation.
- Using the solution of a differential equation to solve problems.
- Finding and using derivatives of functions that involve hyperbolic or inverse trigonometric functions.
- Using L'Hopital's rule to evaluate limits.
- Finding the local maximums and local minimums of a function.
- Setting up and solving optimization word problems (usually involving a constraint).
- Classifying local maximums and minimums using the first derivative.
- Classifying local maximums and minimums using the second derivative.

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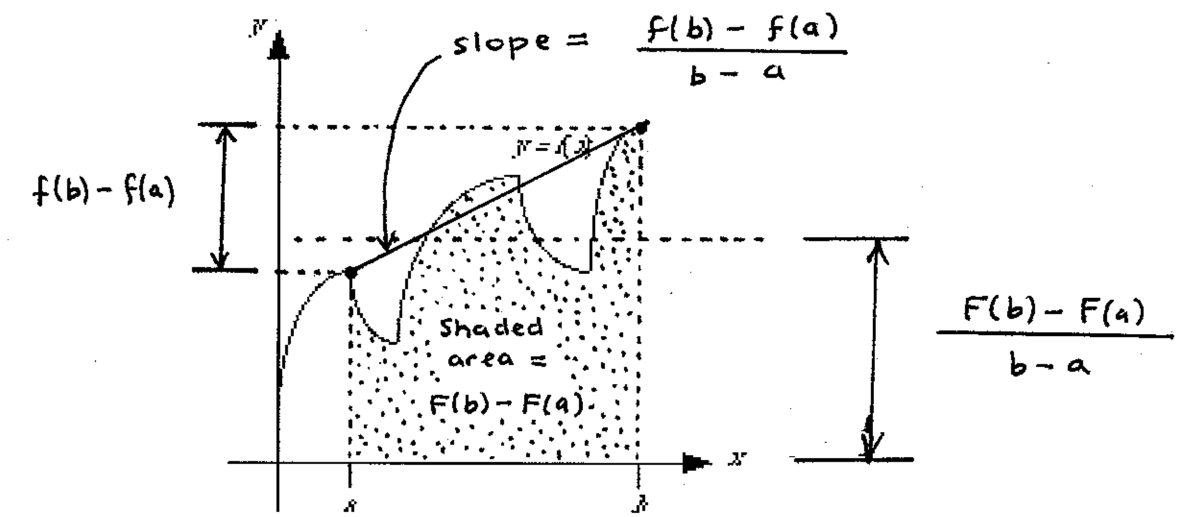
- Finding the global (or absolute) maximum and minimum values of a function when x is confined to a closed interval.
- Sketching graphs of functions based on the behavior of their first and second derivatives.
- Finding solutions to equations using Newton's method.
- Finding anti-derivatives for functions by reversing derivative rules. (Remember the +C.)
- Using anti-derivatives to solve applied physics problems (e.g. distance, velocity, acceleration).
- Left-hand and right-hand Riemann sums (including the "best estimate" which is the average of the left and right-hand sums).
- Midpoint sums.
- Approximating definite integrals using Riemann sums graphically.
- Approximating definite integrals using Riemann sums on a calculator.
- Using area formulas to evaluate or approximate definite integrals and area under a curve.
- Using anti-derivatives to evaluate definite integrals and area under a curve.
- Interpreting the meaning of a definite integral or area under the graph in a "real world" problem (including finding the units of the area).
- Evaluating and graphing functions defined by integration.
- Calculating derivatives for functions defined by integration.
- Approximating function values using Euler's method.
- The technique of u-substitution.
- Integration by parts.

1. A function $F(x)$ has derivative $f(x)$. The derivative function is graphed below.



Represent the following quantities on the graph:

(a) $f(b) - f(a)$.

(b) $\dfrac{f(b) - f(a)}{b - a}$.

(c) $F(b) - F(a)$.

(d) $\dfrac{F(b) - F(a)}{b - a}$.

SOLUTIONS

2. A common article of faith among some business people is that,

Maximum profit occurs when marginal revenue equals marginal cost.

If q items are produced, the cost and revenue can be denoted by $C(q)$ and $R(q)$.

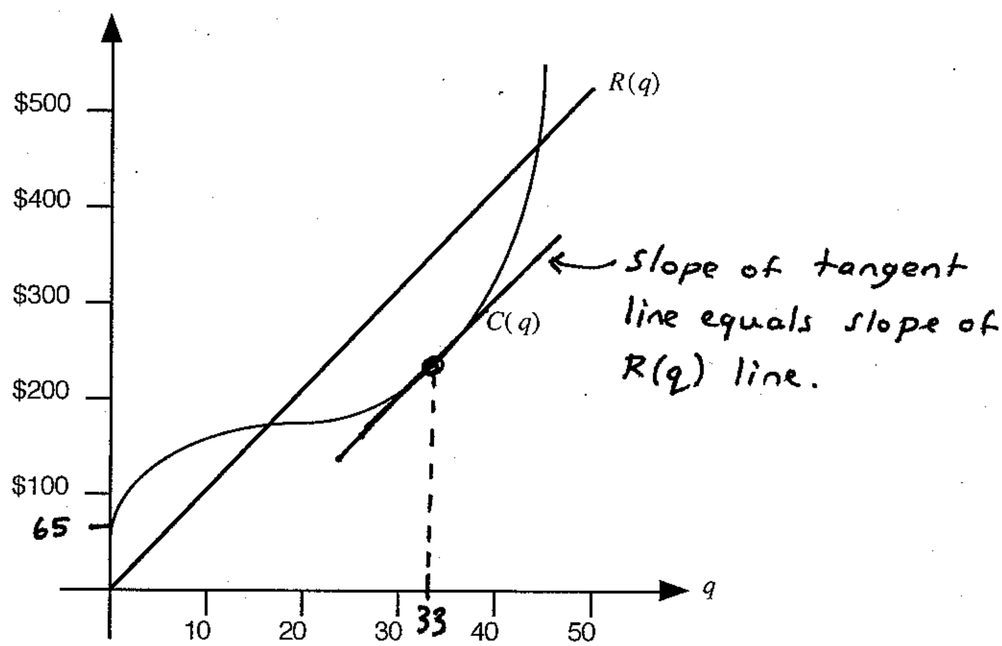
(a) Use calculus to explain why the article of faith is reasonable. Be careful to explain why business people should use caution when using this article of faith.

Business people are usually interested in maximizing profit, $P(q) = R(q) - C(q)$. The maximum value of profit will be at a point where $P'(q) = R'(q) - C'(q) = 0$. At a point where $P'(q) = 0$ we have:

$$\text{Marginal revenue} = R'(q) = C'(q) = \text{Marginal cost.}$$

Caution should be exercised as $R'(q) = C'(q)$ is also true at points where profit is minimized.

(b) Using the graphs given below, estimate how many items should be produced to maximize profits.



For maximum profits, $q \approx 33$.

(c) From the graphs above, what is the *fixed cost*? What is the *fixed revenue*?

The fixed cost is approximately \$65.

The fixed revenue is zero.

SOLUTIONS

3. Find the most general anti-derivative for each of the functions given below, and use the anti-derivative to find the area indicated.

(a) $f(t) = e^{2t} + t^2 - \frac{1}{t}$. Area between $t = 1$ and $t = 4$.

$$F(t) = \frac{1}{2}e^{2t} + \frac{1}{3}t^3 - \ln(|t|) + C$$

$$\text{Area} = F(4) - F(1)$$

$$\approx 1506.3982$$

(b) $u(x) = \cos(x) \cdot e^{\sin(x)} + x^2 \cdot e^{x^3}$. Area between $x = 0$ and $x = 2$.

$$U(x) = e^{\sin(x)} + \frac{1}{3}e^{x^3} + C$$

$$\text{Area} = U(2) - U(0)$$

$$\approx 994.8019$$

(c) $z(t) = \frac{1}{1+t^2} - \frac{1}{\sqrt{1-t^2}}$. Area between $t = 0$ and $t = 0.5$.

$$Z(t) = \tan^{-1}(t) - \sin^{-1}(t) + C$$

$$\text{Area} = Z(0.5) - Z(0)$$

$$\approx -0.05995$$

SOLUTIONS

4. While "makin' their way, the only way they know how," a pair of motorists were followed by a law enforcement officer. The officer's radar equipment took a speed reading every 30 seconds, and printed out the information, as shown below.

Time (seconds)	30	60	90	120	150	180	210	240
Speed (miles per hour)	57	63	64	69	71	69	68	63

(a) Find an upper and a lower estimate for the distance covered by the motorists while the law enforcement officer follows them.

30 seconds $= \frac{1}{120}$ hours.

Upper estimate $= \frac{1}{120}(63+64+69+71+71+69+68) = 3.958$ miles

Lower estimate $= \frac{1}{120}(57+63+64+69+69+68+63) = 3.775$ miles

(b) What is you *best* estimate for the distance covered?

Best estimate $= \dfrac{3.958 + 3.775}{2} = 3.866$ miles

(c) When you calculate the "area" under a velocity-time graph, the result is not usually the total distance traveled. Describe circumstances under which the area under the velocity-time graph would and would not give the total distance traveled.

<u>"Area" would give total distance:</u>

If the velocity curve always stays above the t-axis.

<u>"Area" would not give total distance:</u>

If the velocity curve spends time above and below the t-axis.

SOLUTIONS

5. To convert a temperature in degrees centigrade to a temperature in degrees Fahrenheit, one can follow the following procedure :

1. Multiply the centigrade temperature by 9.
2. Divide the result of step 1 by 5.
3. Add 32 to the result of step 2.

(a) Let x be the temperature in degrees centigrade. Find a formula for $F(x)$, the temperature in degrees Fahrenheit.

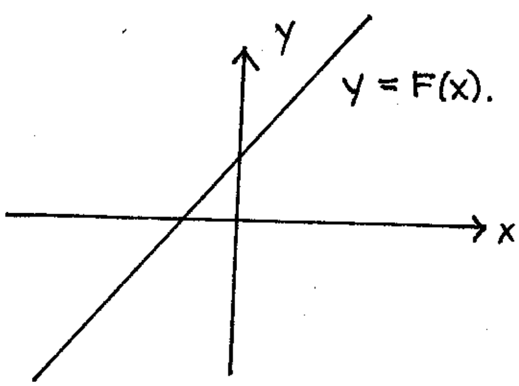
$$F(x) = \frac{9}{5}x + 32$$

(b) Explain the meaning of $F^{-1}(x)$ in practical terms.

$F^{-1}(x)$ = temperature in °C that is equivalent to x °F.

(c) Does $F(x)$ have an inverse ? Either explain why not or find a formula for $F^{-1}(x)$.

Yes. The graph $y = F(x)$ passes the horizontal line test. To find the formula:



$$y = \frac{9}{5}x + 32$$

$$y - 32 = \frac{9}{5}x$$

$$\frac{5}{9}(y - 32) = x$$

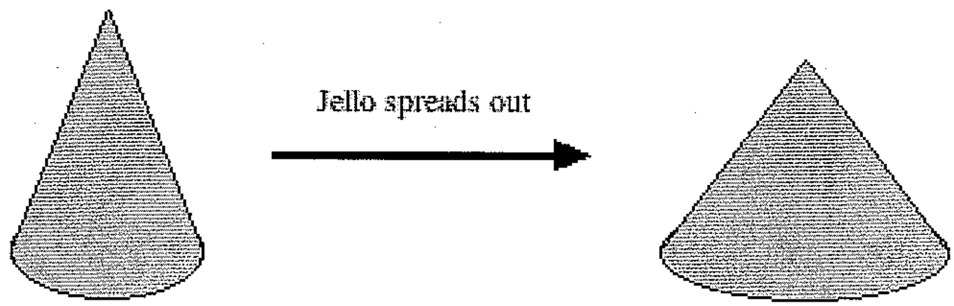
So:

$$F^{-1}(x) = \frac{5}{9}(x - 32).$$

SOLUTIONS

6. The volume of a cone of radius r and height h is given by: $V = \frac{1}{3}\pi r^2 h$.

A cone made out of Jell-O starts out with a radius of 2 inches and a height of 3 inches. However, the Jell-O starts to spread out as illustrated below. When the radius is 3 inches, the rate at which the radius is increasing is 0.2 inches per minute. How fast is the height of the Jell-O changing?



We want: $\frac{dh}{dt}$

We have: $\frac{dr}{dt} = 0.2$ inches / minute

when $r = 3$ inches.

The volume of Jell-O is $V = \frac{1}{3}\pi(2^2)(3)$

$= 4\pi$ cubic inches.

$$\frac{dV}{dt} = \frac{2}{3}\pi r h \cdot \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt} = 0$$ as V is constant.

So: $$\frac{dh}{dt} = \frac{-\frac{2}{3}\pi r h}{\frac{1}{3}\pi r^2} \cdot \frac{dr}{dt}$$

When $r = 3$, $$h = \frac{4\pi}{\frac{1}{3}\pi(3^2)} = \frac{4}{3}$$ inches.

$$\frac{dh}{dt} = \frac{-\frac{2}{3}\pi(3)\left(\frac{4}{3}\right)}{\frac{1}{3}\pi(3^2)}(0.2) \approx -0.177 \text{ inches/minute.}$$

SOLUTIONS

7. The life of a foraging squirrel is not an easy one - it is full of complicated decisions. One decision that squirrels have to make is how much energy they should spend on foraging. In order to find nuts to eat, a squirrel has to expend energy. The number of calories (i.e. $C(q)$) that the squirrel has to expend is given by the function,

$$C(q) = 10 + 0.06q^3.$$

On the other hand, the squirrel gets 30 calories from every nut that it finds.

(a) Find a formula that gives the net energy that the squirrel obtains if it finds and eats q nuts.

$$E(q) = 30q - C(q)$$
$$= 30q - 10 - 0.06q^3$$

(b) If the squirrel is trying to get as much energy as it possibly can (e.g. it might be preparing for the lean months of winter), how many nuts should the squirrel try to collect each day?

$$E'(q) = 30 - 0.18q^2 = 0$$
$$q^2 = \frac{30}{0.18}$$
$$q = 12.9099 \approx 13 \text{ nuts.}$$

To see this is a maximum, use the Second Derivative Test.

$$E''(q) = -0.36q$$
$$E''(12.9099) \approx -4.6475 < 0.$$

SOLUTIONS

8. A curve is described by the equation given below.

$$x^2 + 2x \cdot y + 3y^2 = 2$$

(a) Find a formula for $\frac{dy}{dx}$.

$$2x + 2y + 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x - 2y}{2x + 6y}$$

(b) Verify that the point $(x, y) = (0, \sqrt{\frac{2}{3}})$ lies on the curve, and calculate the equation of the tangent line to this point on the curve.

$$0^2 + 2(0)\left(\sqrt{\tfrac{2}{3}}\right) + 3\left(\sqrt{\tfrac{2}{3}}\right)^2 = 3 \cdot \tfrac{2}{3} = 2.$$

(c) Are there any points on the curve where the tangent line is horizontal? Either explain why not or else find the x- and y-coordinates of one point on the curve where the tangent line is horizontal.

Tangent line is horizontal when $\frac{dy}{dx} = 0$, i.e. $y = -x$.

Substitute this into the equation for the curve:

$$x^2 + 2x(-x) + 3(-x)^2 = 2x^2 = 2$$

So $x = \pm 1$ and the points are: $(1, -1)$ and $(-1, 1)$.

(d) Are there any points on the curve where the tangent line is vertical? Either explain why not or else find the x- and y-coordinates of one point on the curve where the tangent line is vertical.

Tangent line is vertical when $\frac{dy}{dx}$ is undefined, i.e. $y = -\frac{1}{3}x$. Substitute this into the equation for the curve:

$$x^2 + 2x\left(-\tfrac{1}{3}x\right) + 3\left(-\tfrac{1}{3}x\right)^2 = \tfrac{2}{3}x^2 = 2,$$

So $x = \pm\sqrt{3}$ and the points are:

$$\left(\sqrt{3}, -\sqrt{3}/3\right) \text{ and } \left(-\sqrt{3}, \sqrt{3}/3\right).$$

SOLUTIONS

9. For this problem, use the function $g(x)$ defined below.

$$g(x) = (x-2)^2(x-6)^2 + 1$$

(a) Locate the critical points of $g(x)$. You should list both the x- and y- coordinates of the critical points.

$$g'(x) = 2(x-2)(x-6)^2 + 2(x-2)^2(x-6)$$
$$= 2(x-2)(x-6)(x-2+x-6)$$
$$= 4(x-2)(x-6)(x-4)$$

The critical points are:

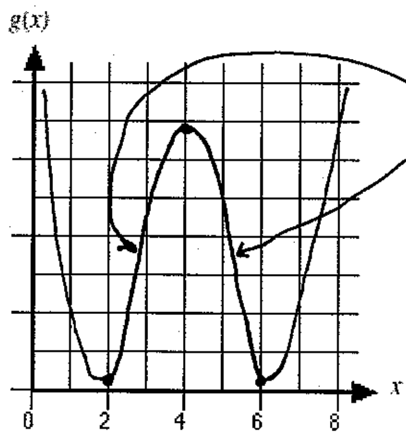
$(2, 1)$ $(4, 17)$ and $(6, 1)$.

(b) Classify the critical points of $g(x)$ - i.e. are the critical points maximums, minimums or neither?

$$g''(x) = 4(x-6)(x-4) + 4(x-2)(x-4) + 4(x-2)(x-6)$$

x	2	4	6
g''	32	−16	32
classification	minimum	maximum	minimum

(c) What happens at inflection points? Sketch a graph of $g(x)$, and indicate where the inflection points of $g(x)$ will be. (There is no need to calculate the precise location of the inflection points.)



The curve changes concavity.

Inflection points are located at the points indicated (approximately).

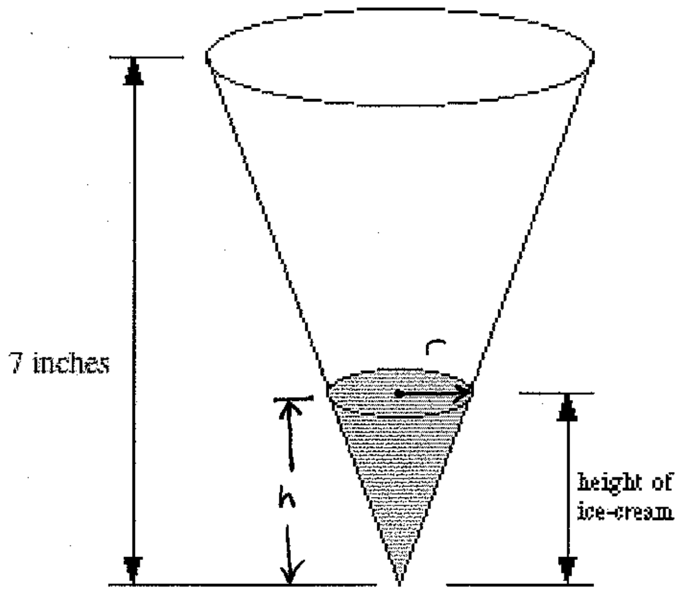
(d) If we just look at the x-values between $x = 3$ and $x = 8$ (inclusive), what is the global maximum and what is global minimum of $g(x)$?

Global maximum: 577 $(x = 8)$

Global minimum: 1 $(x = 6)$.

SOLUTIONS

10. An ice-cream cone is being filled with soft-serve ice cream at a rate of 0.8 cubic inches per second. The cone has a radius of 3 inches and a height of 7 inches. How fast is the level of ice-cream rising when the level of ice cream is 3 inches?



Using Similar Triangles we have:

$$\frac{r}{h} = \frac{3}{7} \quad \text{so}$$

$$r = \frac{3h}{7}.$$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi \left(\frac{3h}{7}\right)^2 h$$

$$= \frac{3\pi}{49} h^3$$

<u>Given:</u> $\frac{dV}{dt} = 0.8$ cubic inches/second.

<u>Want:</u> $\frac{dh}{dt}$ when $h = 3$ inches.

$$\frac{dV}{dt} = \frac{3\pi}{49} 3h^2 \cdot \frac{dh}{dt}, \quad \text{so:}$$

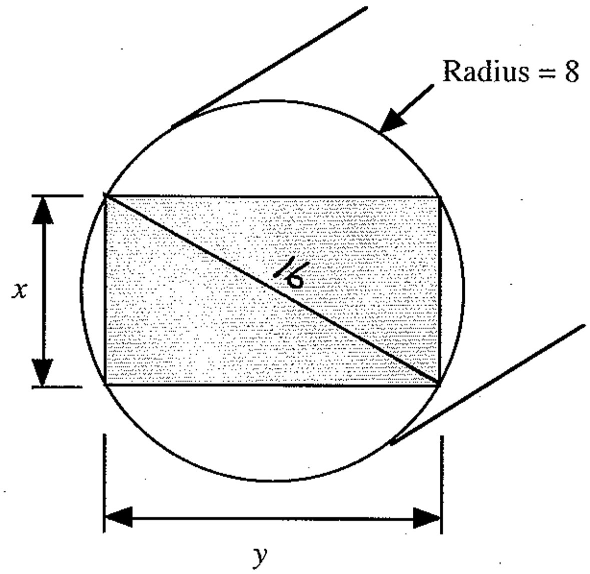
$$\frac{dh}{dt} = \frac{49(0.8)}{(3\pi)3(3^2)} = 0.154046 \text{ inches/second.}$$

SOLUTIONS

11. A sawmill is planning to cut a log into a beam. The log has a cylindrical cross-section with a radius of eight (8) inches. The beam will have a rectangular cross-section with a height of x and a width of y. The strength, S, of the finished beam will be given by the formula:

$$S = 5xy^2.$$

What values of x and y should be used for the beam if the people at the sawmill wish to make the strongest possible beam? Circle your final answer.



Using the Theorem of Pythagoras:

$$x^2 + y^2 = 16^2$$

so

$$y^2 = 16^2 - x^2$$

and

$$S = 5x(16^2 - x^2)$$
$$= 1280x - 5x^3$$

$$\frac{dS}{dx} = 1280 - 15x^2 = 0$$

$$x = \sqrt{\frac{1280}{15}} = 9.2376 \text{ inches.}$$

To verify that this maximizes S:

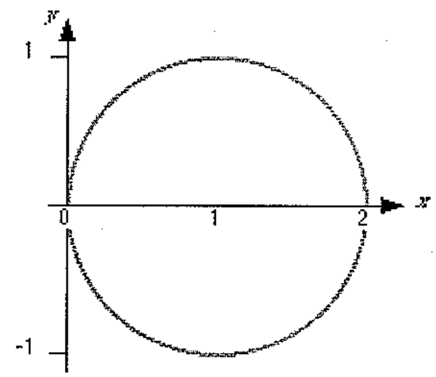
$$\frac{d^2S}{dx^2} = -30x < 0 \quad \text{for } x = 9.2376.$$

Values of x and y are:

$x = 9.2376$ inches $\quad y = 13.0639$ inches.

SOLUTIONS

12. The diagram given below shows a curve in the xy- plane.



(a) The x- and y-coordinates of the points that lie on this curve satisfy the equation:

$$y^2 + (x-1)^2 = 1.$$

Is it possible to find a function $f(x)$ so that the curve shown above is the graph of $y = f(x)$? If so, find an equation for $f(x)$. If not, explain why not.

No. The graph fails the vertical line test.

(b) Find an equation for the derivative of y with respect to x - that is, an equation for $\frac{dy}{dx}$.

$$2yy' + 2(x-1) = 0 \qquad \frac{dy}{dx} = y' = \frac{-2(x-1)}{2y} = \frac{1-x}{y}$$

(c) The portion of the curve that lies above the x-axis can be represented by the equation:

$$y = \sqrt{1-(x-1)^2}.$$

Use this explicit equation for y as a function of x to find an equation for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{-(x-1)}{\sqrt{1-(x-1)^2}}$$

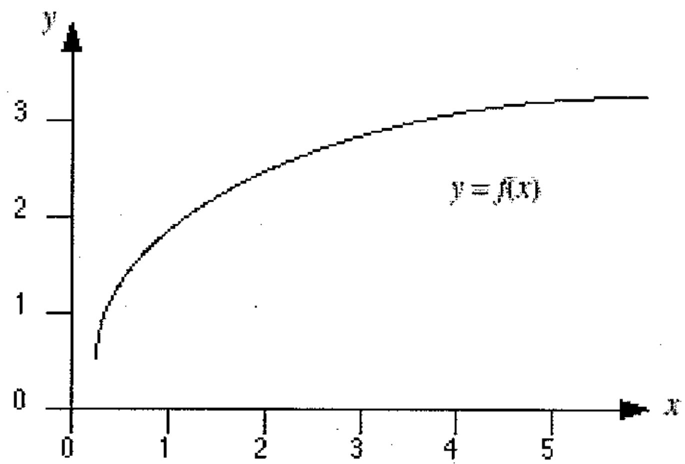
(d) In Parts (b) and (c) of this problem, you found an equation for $\frac{dy}{dx}$ in two different ways. Chances are, the two equations that you found for $\frac{dy}{dx}$ look quite different as well. Are these two equations really the same or are they different in some important way?

They are the same so long as $y > 0$. When $y > 0$, $y = \sqrt{1-(x-1)^2}$ so:

$$\frac{-(x-1)}{\sqrt{1-(x-1)^2}} = \frac{1-x}{y}.$$

SOLUTIONS

13. Consider the function $y = f(x)$ graphed below. For each of the following pairs of numbers, decide which is larger.



(a) $f(3)$ or $f(4)$?

$$f(3) < f(4)$$

(b) $f(3) - f(2)$ or $f(2) - f(1)$?

$$f(3) - f(2) < f(2) - f(1)$$

(c) $\dfrac{f(2) - f(1)}{2 - 1}$ or $\dfrac{f(3) - f(1)}{3 - 1}$?

$$\frac{f(3) - f(1)}{3 - 1} < \frac{f(2) - f(1)}{2 - 1}$$

(d) $f'(1)$ or $f'(4)$?

$$f'(4) < f'(1)$$

SOLUTIONS

14. In this problem, $f(x)$ and $g(x)$ are functions that have derivatives. All that you can assume about them is

- $f'(2) = 7$
- $f(2) = 2$
- $g'(2) = -4$
- $g(2) = 18.$

Use the information given about $f(x)$ and $g(x)$ to calculate the derivatives of the functions given below.

(a) $p'(2)$, where $p(x) = \dfrac{f(x)}{g(x)}$.

$$p'(2) = \frac{g(2)\,f'(2) - f(2)\,g'(2)}{g(2)^2} = \frac{(18)(7) - (2)(-4)}{18^2}$$

$$\approx 0.41358$$

(b) $q'(2)$, where $q(x) = f(x)\cdot[g(x)]^2$.

$$q'(2) = f'(2)\cdot g(2)^2 + f(2)\cdot 2\cdot g(2)\cdot g'(2)$$

$$= (7)(2^2) + (2)(2)(18)(-4)$$

$$= -260$$

(c) $j'(2)$, where $j(x) = [f(x) + g(x)]^4$.

$$j'(2) = 4(f(2) + g(2))^3\cdot(f'(2) + g'(2))$$

$$= 4(2 + 18)^3\cdot(7 + -4)$$

$$= 96\,000$$

(d) $m'(2)$, where $m(x) = g(x)\cdot\ln(f(x))$.

$$m'(2) = g'(2)\cdot\ln(f(2)) + g(2)\cdot\frac{f'(2)}{f(2)}$$

$$= (-4)\cdot\ln(2) + (18)\cdot\frac{7}{2}$$

$$\approx 60.2274$$

SOLUTIONS

15. Organic matter contains a radioactive isotope of carbon, known as carbon-14. The half life of carbon-14 is 5730 years. A 100g sample of fresh organic matter will normally contain 0.0001 μg of carbon-14. In this problem, you will examine some of the ways in which the smallest mistakes can drastically alter results when exponential functions and logarithms are involved.

(a) Find an equation to describe the amount of carbon-14 (in micrograms) that remains in a 100g sample of organic matter after T years.

$$B = \left(\tfrac{1}{2}\right)^{1/5730} \doteq 0.9998790392$$

$C(T)$ = micrograms of C-14 that remain after T years.

$$C(T) = 0.0001 \cdot (0.9998790392)^T$$

(b) Suppose that a 100g sample of organic matter contains 0.0000327 μg of carbon-14. How old is the organic matter?

$$0.0000327 = 0.0001 \cdot (0.9998790392)^T$$

$$T = \frac{\ln\left(\frac{0.0000327}{0.0001}\right)}{\ln(0.9998790392)} \approx 9240.41096 \text{ years}$$

(c) Now, suppose that you misread the number from Part (b) and used the figure of 0.0000372 μg of carbon-14 instead. How far off would the age of the organic matter be?

$$T = \frac{\ln\left(\frac{0.0000372}{0.0001}\right)}{\ln(0.9998790392)} \approx 8174.56248 \text{ years}$$

$$\text{Difference} = 9240.41096 - 8174.56248$$
$$= 1065.84848 \text{ years.}$$

(d) Lastly, suppose that your little buddy Barry calculated the age of a sample of organic matter and determined that it was 2000 years old. Looking over Barry's work, you notice that he incorrectly remembered the half-life of carbon-14 to be 5370 years and used this figure in his calculations. What is the correct age of the organic matter?

$$B_{Barry} = \left(\tfrac{1}{2}\right)^{1/5370} \doteq 0.9998709306$$

C = amount of C-14 in sample.

$$C = 0.0001\,(0.9998709306)^{2000} \doteq 7.724751998 \times 10^{-5}\ \mu g.$$

$$\text{Correct age} = \frac{\ln\left(\frac{C}{0.0001}\right)}{\ln(0.9998790392)} \doteq 2134.0778 \text{ years.}$$

SOLUTIONS

16. In this problem the function f will always be the function defined by the equation given below.

$$f(x) = \frac{1}{1+e^{-x}}.$$

(a) Determine the intervals on which f is an increasing function and the intervals on which f is a decreasing function.

$$f'(x) = -(1+e^{-x})^{-2} \cdot (-e^{-x})$$

$$= \frac{e^{-x}}{(1+e^{-x})^2}$$

This is always positive, so $f(x)$ is increasing for all real numbers, x.

(b) Is the inverse of f a function in its own right? If so, find a formula for the inverse of f.

Yes it is. As $f(x)$ is always increasing it passes the vertical line test.

$$y = \frac{1}{1+e^{-x}}$$

$$\frac{1}{y} = 1 + e^{-x}$$

$$\frac{1}{y} - 1 = e^{-x}$$

$$\ln\left(\frac{1}{y} - 1\right) = -x$$

$$x = f^{-1}(y) = -\ln\left(\frac{1}{y} - 1\right).$$

SOLUTIONS

17. In 1975 the population of Mexico was about 84 million. In this problem, T will always represent the number of years since 1975 and $M(T)$ will always represent the population of Mexico (in units of millions of people. The equation for $M(T)$ is:

$$M(T) = 84 \cdot (1.026)^T.$$

In this problem $U(T)$ will always represent the population of the United States (in units of millions of people). The equation for $U(T)$ is:

$$U(T) = 250 \cdot (1.007)^T.$$

(a) Find an equation for the instantaneous rate of change of the population of Mexico, $M'(T)$.

$$M'(t) = 84 \cdot \ln(1.026) \cdot (1.026)^T$$

(b) How quickly was the population of Mexico changing in 1990 (when $T = 15$)? Give a practical interpretation of this number that someone who had not studied calculus could understand.

$M'(15) \approx 3.16867$ million people per year.

During the year 1990, the population of Mexico increased by approximately 3,168,670 people.

(c) Is it ever possible that the population of Mexico will equal the population of the United States? If you believe that this could happen, calculate the year when the two populations will be equal. If you believe that this cannot happen explain why.

Yes, it is possible. $84(1.026)^T = 250 \cdot (1.007)^T$

$$T = \frac{\ln\left(\frac{84}{250}\right)}{\ln\left(\frac{1.007}{1.026}\right)} \approx 58.34776$$

The year will be 2033 or 2034.

SOLUTIONS

18. In this problem, the function $f(x)$ is always the function defined in Figure 1 below.

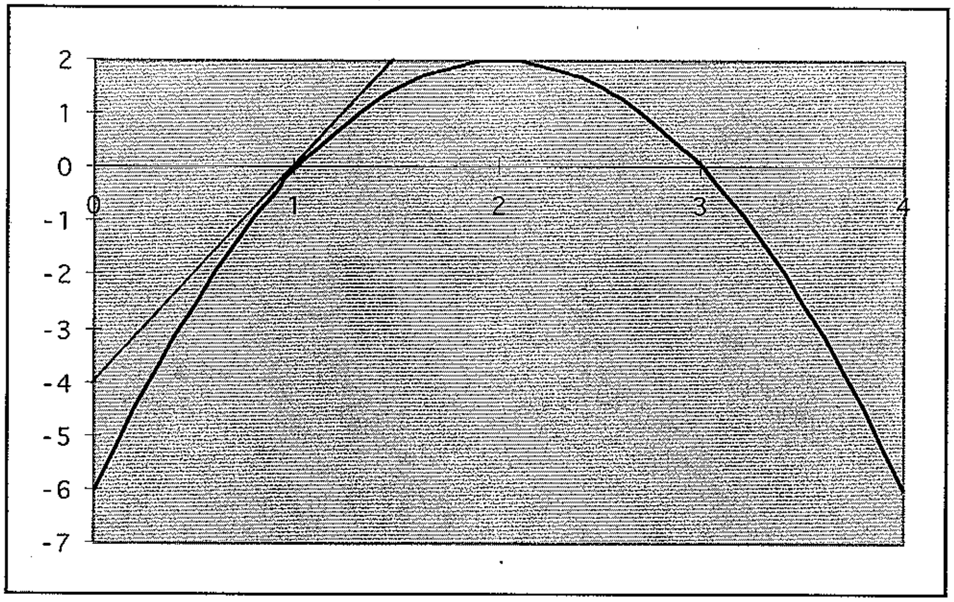


Figure 1: Graph defining $f(x)$.

In this problem, the function $g(x)$ is always the function defined by the equation: $g(x) = 10^x$.

Use the functions $f(x)$ and $g(x)$ to calculate the values of the following derivatives:

(a) $f'(2)$.

$$f'(2) = 0 \qquad f'(1) \approx 4$$

$$f''(2) \approx \frac{-4 - 4}{3 - 1} = -4$$

(b) $k'(2)$ where $k(x) = \frac{f'(x)}{g'(x)}$.

$$k'(2) = \frac{g'(2) \cdot f''(2) - f'(2)\, g''(2)}{g'(2)^2}$$

$$\approx \frac{\ln(10) \cdot 10^2 \cdot (-4) - (0) \cdot g''(2)}{((\ln(10)) \cdot 10^2)^2} \doteq -0.01737$$

(c) $m'(1)$ where $m(x) = f(x) \cdot g(x)$.

$$m'(1) = f'(1) \cdot g(1) + f(1) \cdot g'(1)$$

$$= (4) \cdot 10^1 + (0) \cdot g'(1)$$

$$= 40.$$

SOLUTIONS

19. In this problem, the functions f, g and h will always refer to:

- $f(x) = x^2 - 4$
- $g(x) = x^2 + 4$
- $h(x) = x + 5$.

Using these three functions as building blocks, create equations for functions that have the properties describe below.

(a) The function is defined for all real numbers. The function has a horizontal asymptote at $y = 0$ and an x-intercept at $x = -5$.

$$\frac{h(x)}{g(x)}$$

(b) The function has x-intercepts at $x = -5$, $x = -2$ and $x = 2$. The function does not have any horizontal or vertical asymptotes.

$$f(x) \cdot h(x)$$

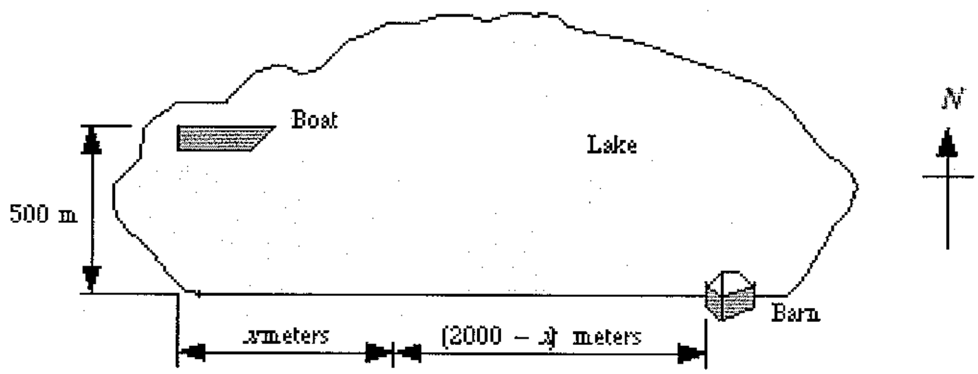
(c) The function has no x-intercepts. The function has a horizontal asymptote at $y = 0$. The function has no vertical asymptotes.

$$\frac{1}{g(x)}$$

SOLUTIONS

20. Under some conditions, it is easier for a bird to fly over land than it is for the bird to fly over water. This is because the temperature of a large body of water (like a lake or the ocean) is much more stable than the temperature of the land. Land tends to heat up more than water does, creating thermal updrafts that make it easier for the birds to stay aloft. One reference[1] suggests that when flying over water, homing pigeons have to expend 30 joules for every one meter that they fly. The same pigeons only have to expend 10 joules for every one meter that they fly over land.

The diagram below shows the set-up for an annual race held by pigeon enthusiasts. The pigeons are released from a boat in the middle of a lake, and the finish line is a barn on the edge of the lake. The pigeons fly at the same speed no matter whether they fly over water or over land.



When released from the boat, the pigeons normally fly in a diagonal line towards the shore. When they reach the shore, the pigeons normally fly along the shore until they reach the barn, where they land.

(a) Find an equation that describes the amount of energy expended by a pigeon during the race. (The variable in your equation should be x.)

$$E = 10\cdot(2000 - x) + 30\cdot\sqrt{500^2 + x^2}$$

(b) Generally speaking, most animals try to conserve energy when possible. Where should the pigeon come ashore in order to expend the least amount of energy during the race?

$$E' = -10 + \frac{30x}{\sqrt{500^2 + x^2}} = 0$$

$9x^2 = 500^2 + x^2$ solving for x: $x = \sqrt{\frac{500^2}{8}} = 176.7767$ m

(c) Recall that pigeons fly at the same speed regardless of whether they are flying over land or water. This is about 5 meters per second. Find an equation that gives the amount of time that a pigeon needs to complete the race. (The variable in your equation should be x.)

$$T = \frac{2000 - x + \sqrt{500^2 + x^2}}{5}$$

(d) Where should the pigeon come ashore in order to complete the race in the least amount of time?

$$T' = \frac{1}{5}\left(-1 + \frac{x}{\sqrt{500^2 + x^2}}\right) = 0$$ No solutions.

Endpoints:

x	0	2000
T	500	412.31

Come ashore at the barn.

[1] Deborah Hughes-Hallet, Andrew Gleason, et al. "Calculus." New York: John Wiley and Sons, 1994.

SOLUTIONS

21. In this problem, f and g are differentiable functions. All that you can assume about them is the information given in the table below.

f(1) = 2	g(1) = 4	$f'(1) = 3$	$g'(1) = 7$
f(2) = 6	g(2) = −1	$f'(2) = 8$	$g'(2) = 15$
f(3) = 0	g(3) = 1	$f'(3) = 6$	$g'(3) = 17$
f(4) = 8	g(4) = −5	$f'(4) = 9$	$g'(4) = 21$

Use the information given in the table to evaluate the following:

(a) $a'(2)$ where $a(x) = f(x)\cdot g(x)$.

$$a'(2) = f'(2)\cdot g(2) + f(2)\cdot g'(2) = (8)(-1) + (6)(15)$$
$$= 82$$

(b) $b'(1)$ where $b(x) = \dfrac{f(x)}{g(x)}$.

$$b'(1) = \frac{f'(1)\cdot g(1) - g'(1)\cdot f(1)}{g(1)^2} = \frac{(3)(4) - (7)(2)}{16}$$
$$= -0.125$$

(c) $c'(3)$ where $c(x) = f(x) + g(x)$.

$$c'(3) = f'(3) + g'(3) = 6 + 17 = 23.$$

(d) $d'(1)$ where $d(x) = f(g(x))$.

$$d'(1) = f'(g(1))\cdot g'(1) = f'(4)\cdot(7) = (9)(7) = 63.$$

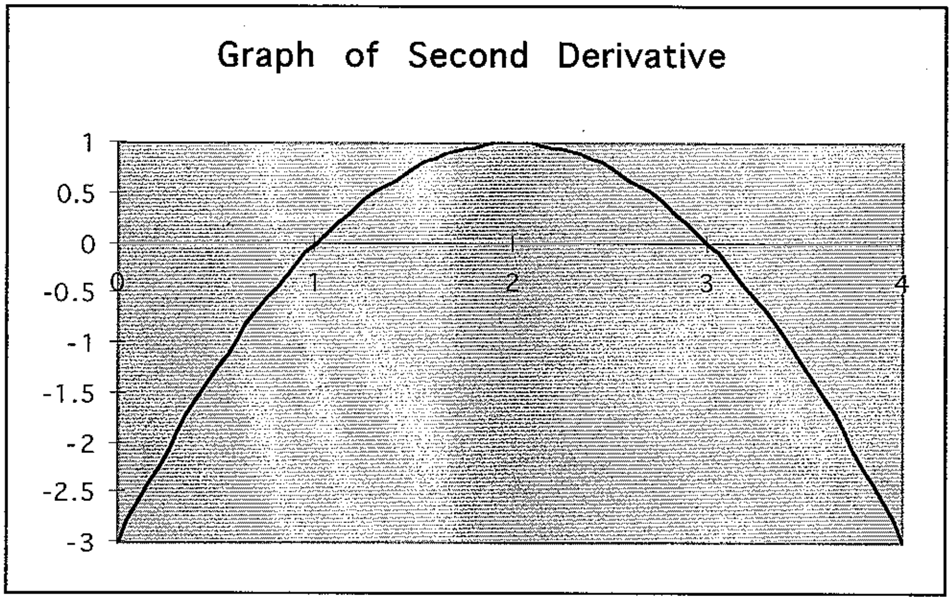
(e) $k'(1)$ where $k(x) = g(f(x))$.

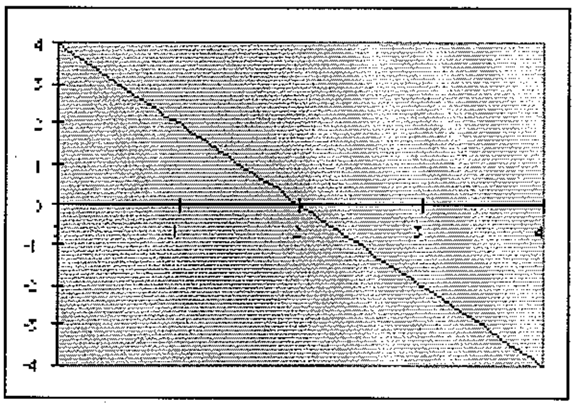
$$k'(1) = g'(f(1))\cdot f'(1) = g'(2)\cdot f'(1) = (5)(3) = 45$$

(f) $k'(2)$ where $k(x) = g(f(x))$.

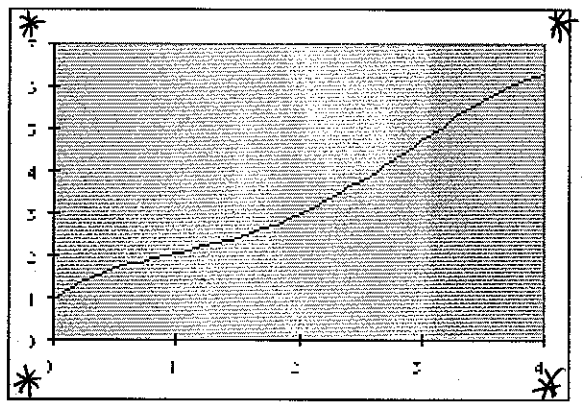
$$k'(2) = g'(f(2))\cdot f'(2) = g'(6)\cdot(8)$$

22. The graph given below is the graph of the *second derivative* of a function. Explain which of the graphs (a, b, c or d) given below could be the graph of the original function.

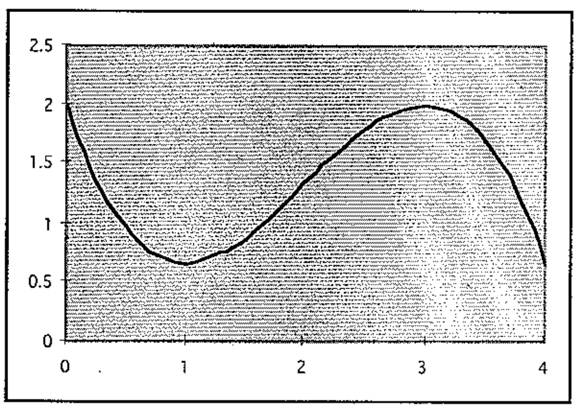




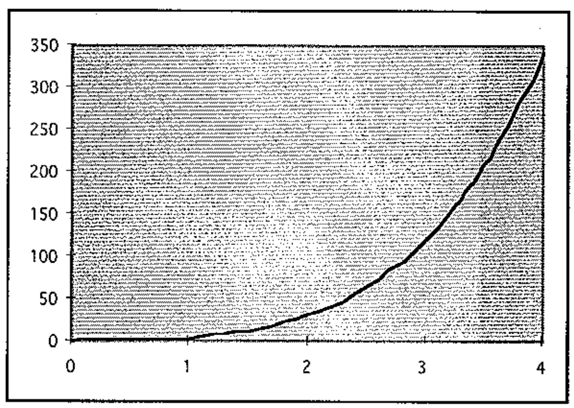
Graph A.



Graph B.



Graph C.



Graph D.

Graph B is probably the original function. The second derivative graph shows that the concavity of the original function changes at $x = 1$ and $x = 3$. Graph B is the only graph that changes concavity at these points.

SOLUTIONS

23. In this problem, the function $f(x)$ will always refer to the function defined by:

$$f(x) = \frac{1}{x}.$$

(a) Using any short-cuts or differentiation rules that you know, find an equation for the derivative $f'(x)$.

$$f'(x) = \frac{-1}{x^2}$$

(b) Use the formula given for $f(x)$ to create and expression for the difference quotient,

$$\frac{f(x+h) - f(x)}{h}.$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{-1}{(x+h)} + \frac{1}{x}}{h}$$

(c) Simplify the expression that you created in Part (b) until it is possible to cancel out the h in the denominator of the difference quotient.

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h} \cdot \frac{x - (x+h)}{x \cdot (x+h)}$$

$$= \frac{1}{h} \cdot \frac{-h}{x \cdot (x+h)}$$

$$= \frac{-1}{x \cdot (x+h)}, \quad h \neq 0$$

(d) By simplifying the difference quotient as much as possible and taking the limit as $h \to 0$, find an equation for the derivative $f'(x)$.

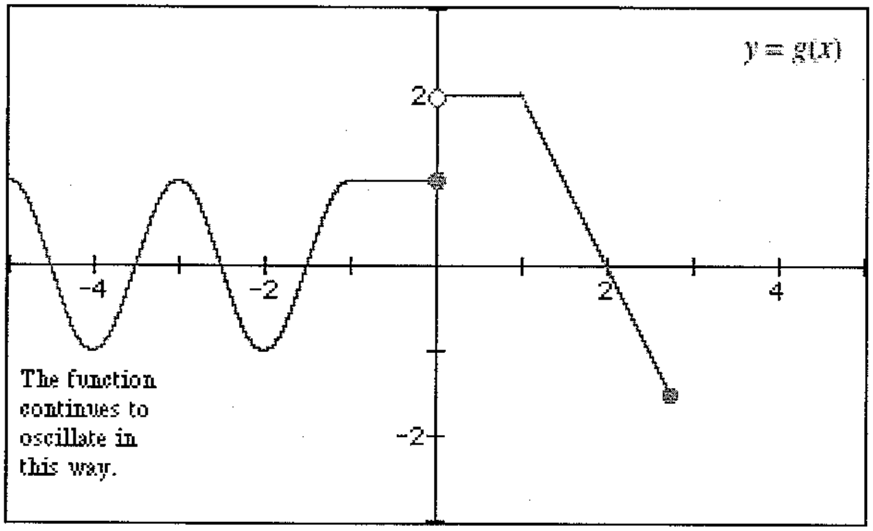
$$f'(x) = \lim_{h \to 0} \frac{-1}{x \cdot (x+h)} = \frac{-1}{x^2}$$

SOLUTIONS

24. In this problem, the function f will always refer to the function defined by the equation:

$$f(x) = \frac{1-x}{1-x^2}$$

and the function g will always refer to the function defined by the graph:



In each of the following cases decide whether or not the limit (possibly a left- or a right-hand limit, check the notation carefully) exists. If you believe that a limit exists, determine its value. If you believe that a limit does not exist, give your reason.

(a) $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \frac{(1-x)}{(1-x)(1+x)} = \lim\limits_{x \to 1} \frac{1}{1+x} = \frac{1}{2}$

(b) $\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} \frac{1}{1+x} = +\infty$

(c) $\lim\limits_{x \to 0} g(x)$ does not exist because $\lim\limits_{x \to 0^+} g(x) = 2$ while $\lim\limits_{x \to 0^-} g(x) = 1.$

(d) $\lim\limits_{x \to 2} \frac{f(x)}{g(x)}$ does not exist because $\lim\limits_{x \to 2^+} \frac{f(x)}{g(x)} = -\infty$ while $\lim\limits_{x \to 2^-} \frac{f(x)}{g(x)} = +\infty.$

SOLUTIONS

25. In this problem the function $f(x)$ will always refer to the function defined by the equation:

$$f(x) = x^2 + 3 \cdot x + 1.$$

(a) Using any short-cuts or differentiation rules that you know, find an equation for the derivative $f'(x)$.

$$f'(x) = 2x + 3$$

(b) Use the formula given for $f(x)$ to create and expression for the difference quotient,

$$\frac{f(x+h) - f(x)}{h}.$$

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 + 3(x+h) + 1 - (x^2 + 3x + 1)}{h}$$

(c) Simplify the expression that you created in Part (b) until it is possible to cancel out the h in the denominator of the difference quotient.

$$\frac{f(x+h) - f(x)}{h} = \frac{x^2 + 2xh + h^2 + 3x + 3h + 1 - x^2 - 3x - 1}{h}$$

$$= \frac{2xh + h^2 + 3h}{h}$$

$$= 2x + h + 3 \quad , \quad h \neq 0$$

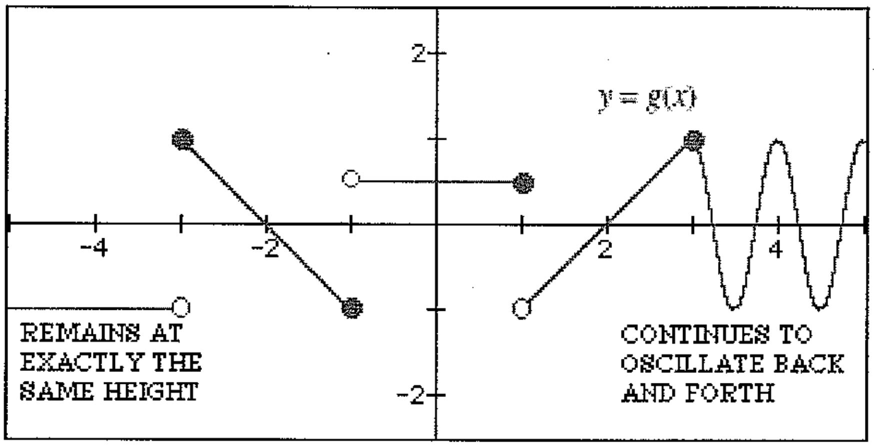
(d) By simplifying the difference quotient as much as possible and taking the limit as $h \to 0$, find an equation for the derivative $f'(x)$.

$$f'(x) = \lim_{h \to 0} 2x + h + 3 = 2x + 3.$$

26. In this problem the function $f(x)$ will always refer to the function defined by the equation:

$$f(x) = x.$$

and the function $g(x)$ will always refer to the function defined by the graph:



Calculate the limits given below. If you believe that the limit exists calculate the value of the limit. If you believe that the limit does not exist, briefly indicate why.

(a) $\lim\limits_{x \to 1} g(x)$ does not exist as $\lim\limits_{x \to 1^-} g(x) = 1/2$ and $\lim\limits_{x \to 1^+} g(x) = -1$.

(b) $\lim\limits_{x \to 2^+} \dfrac{f(x)}{g(x)} = +\infty$

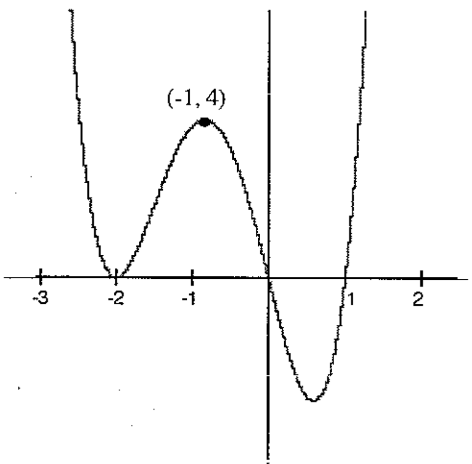
(c) $\lim\limits_{x \to 2} \dfrac{f(x)}{g(x)}$ does not exist because $\lim\limits_{x \to 2^+} \dfrac{f(x)}{g(x)} = +\infty$ while $\lim\limits_{x \to 2^-} \dfrac{f(x)}{g(x)} = -\infty$.

(d) $\lim\limits_{x \to -\infty} \dfrac{g(x)}{f(x)} = 0.$

SOLUTIONS

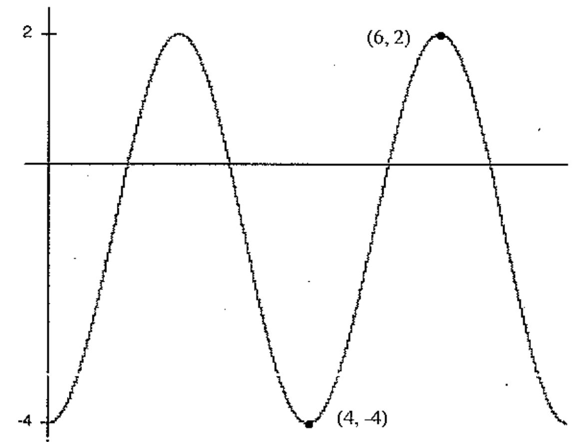
27. Find formulas for each of the following functions. Your formulas should not contain any unspecified constants. You can assume that (b) is part of the graph of a periodic function, (c) is the graph of a power function and that (d) is the graph of an exponential function. Show your work. No work = no credit.

(a)



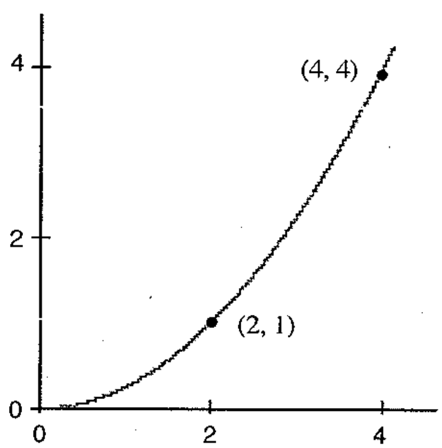
$$y = 2 \cdot (x+2)^2 (x)(x-1)$$

(b)



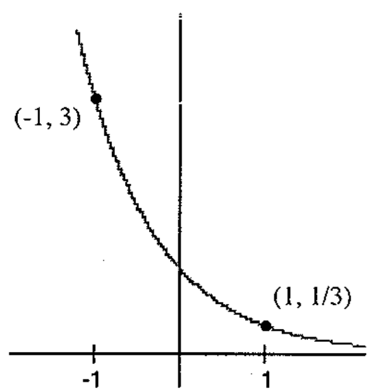
$$y = -3\cos\left(\frac{2\pi}{4}x\right) - 1$$

(c)



$$y = \frac{1}{4}x^2$$

(d)



$$y = 1 \cdot \left(\frac{1}{3}\right)^x$$

SOLUTIONS

28. Find all solutions of the following equations using the laws of logarithms and the laws of exponents. In this problem you should not use your calculator, except for simple arithmetic and to evaluate the values of exponential and logarithmic functions.

(a) $12(1.9)^x = 24.$

$$x = \frac{\ln(24/12)}{\ln(1.9)}$$

$$\approx 1.079914$$

(b) $\log(4x-1)\cdot\log(2x) = 0.$

$$x = 1/2.$$

(c) $x^2\ln(x+1) - x\ln(x+1) = 0.$

$$x = 0, \quad x = 1.$$

(d) $5e^{0.4x} = 20(0.7)^x.$

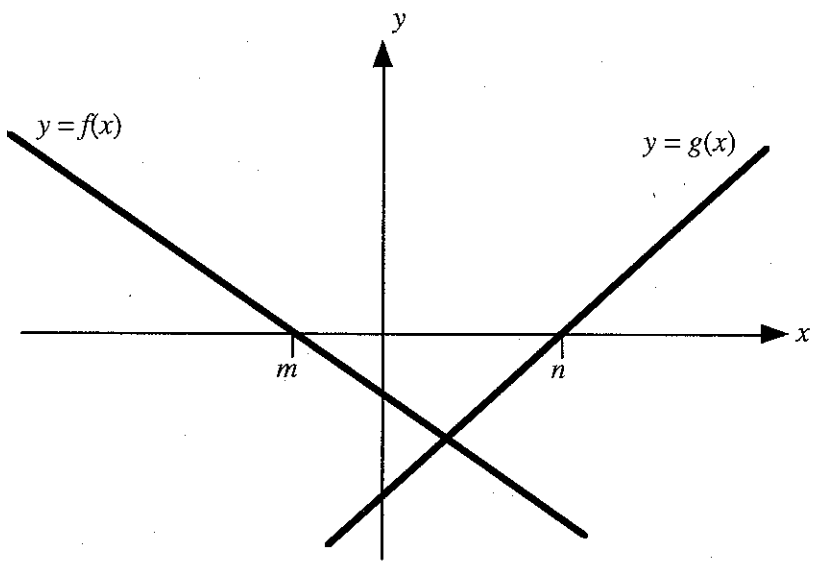
$$\ln(5) + 0.4x = \ln(20) + x\cdot\ln(0.7)$$

$$x = \frac{\ln(20) - \ln(5)}{0.4 - \ln(0.7)}$$

$$\approx 1.832087$$

SOLUTIONS

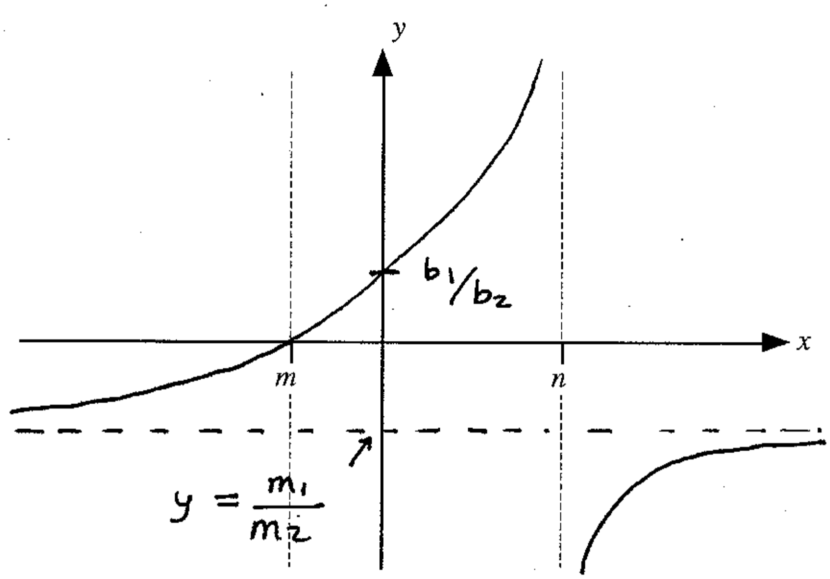
29. The diagram given below shows the graphs of two linear functions, $f(x)$ and $g(x)$. All that you can assume about these linear functions is that the x-intercept of $f(x)$ is located at $x = m$, that the x-intercept of $g(x)$ is located at $x = n$, that the y-intercepts are both negative, and that the domain of each linear function includes all real numbers.



Use the axes given below to sketch an accurate graph of the function $h(x) = \frac{f(x)}{g(x)}$. Carefully label your sketch to indicate the locations of important features of $h(x)$ such as asymptotes and intercepts.

$f(x) = m_1 x + b_1$

$g(x) = m_2 x + b_2$



SOLUTIONS

30. In each case, determine whether the limit exists. If the limit exists, find its value.

(a) $\lim_{x \to 0} \frac{\tan(\pi \cdot x)}{\ln(1+x)} \overset{L.H.}{=} \lim_{x \to 0} \frac{\pi \cdot \sec^2(\pi x)}{\frac{1}{1+x}} = \pi$

(b) $\lim_{x \to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln(x)} \right) = \lim_{x \to 1^+} \frac{x \cdot \ln(x) - x + 1}{(x-1) \cdot \ln(x)}$

$$\overset{L.H.}{=} \lim_{x \to 1^+} \frac{\ln(x) + 1 - 1}{\ln(x) + (x-1) \cdot \frac{1}{x}}$$

$$\overset{L.H.}{=} \lim_{x \to 1^+} \frac{1/x}{1/x + 1/x^2} = \frac{1}{2}$$

(c) $\lim_{x \to \infty} \frac{\ln(\ln(x))}{x \cdot \ln(x)} \overset{L.H.}{=} \lim_{x \to \infty} \frac{\frac{1}{x \cdot \ln(x)}}{\ln(x) + 1} = 0.$