

Handout 20: In-class Review for Exam 3

The topics covered by Exam 3 in the course include the following:

- Finding the local maximums and local minimums of a function.
- Setting up and solving optimization word problems (usually involving a constraint).
- Classifying local maximums and minimums using the first derivative.
- Classifying local maximums and minimums using the second derivative.
- Finding the global (or absolute) maximum and minimum values of a function when x is confined to a closed interval.
- Sketching graphs of functions based on the behavior of their first and second derivatives.
- Finding solutions to equations using Newton's method.
- Finding anti-derivatives for functions by reversing derivative rules. (Remember the +C.)
- Using anti-derivatives to solve applied physics problems (e.g. distance, velocity, acceleration).
- Left-hand and right-hand Riemann sums (including the "best estimate" which is the average of the left and right-hand sums).
- Midpoint sums.
- Approximating definite integrals using Riemann sums graphically.
- Approximating definite integrals using Riemann sums on a calculator.
- Using area formulas to evaluate or approximate definite integrals and area under a curve.
- Using anti-derivatives to evaluate definite integrals and area under a curve.
- Interpreting the meaning of a definite integral or area under the graph in a "real world" problem (including finding the units of the area).
- Evaluating and graphing functions defined by integration.
- Calculating derivatives for functions defined by integration.
- Approximating function values using Euler's method.
- The technique of u-substitution.

1. Find all values of x for which the function $F(x) = \int_0^x \frac{1}{1+t^2} dt$ is *concave up*.

$$F'(x) = \frac{1}{1+x^2} \quad F''(x) = \frac{-2x}{(1+x^2)^2}$$

$F(x)$ is concave up when $F''(x) > 0$. Solving the inequality:

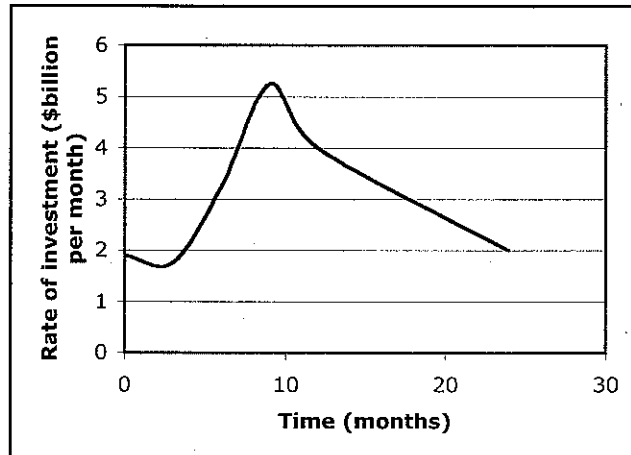
$$F''(x) = \frac{-2x}{(1+x^2)^2} > 0$$

gives: $x < 0$.

SOLUTIONS

2. The graph¹ given below shows the **rate** at which venture capitalists invested money in U.S. software firms companies in the period January 1999-December 2001. The units of the vertical axis are billions of dollars per month and the units of the horizontal axis are months.

In this problem let T represent the number of months since the start of January 1999 (so that $T = 0$ at the beginning of January 1999) and $r(T)$ the **rate** at which venture capitalists were investing money in U.S. software firms.



RATE at which venture capitalists invested money in US software firms.

- (a) What are the units of the area under this curve?

Billions of dollars.

- (b) What does the area under the curve between $T = 1$ and $T = 12$ represent (in terms of money, investments, software firms, etc.)?

The amount of money invested in software firms (in units of billions of dollars) by venture capitalists between February and December, 1999.

- (c) Does the graph given above suggest anything about how venture capitalists now feel about software as an investment opportunity? Explain what, and how you can tell this from the graph.

The graph was trending downwards at its end suggesting that venture capitalists are now not very interested in investing money in software companies.

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¹ Source: VentureOne Financial Analysts, Inc.

SOLUTIONS

- (d) The curve shown on the previous page can be approximated by the quartic equation:

$$r(T) = 0.001 \cdot T^4 - 0.034 \cdot T^3 + 0.39 \cdot T^2 - 1.02 \cdot T + 1.94.$$

Find an equation for the anti-derivative, $R(T)$, of this quartic equation. Your anti-derivative should contain one unspecified constant.

$$R(T) = \frac{0.001}{5} T^5 - \frac{0.034}{4} T^4 + \frac{0.39}{3} T^3 - \frac{1.02}{2} T^2$$

$$+ 1.94 T + C$$



unspecified constant.

- (e) Calculate the numerical value of the quantity:

$$\int_0^{10} r(T) \cdot dT.$$

What are the units of this quantity?

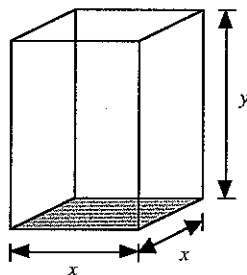
$$\int_0^{10} r(T) dT = R(10) - R(0)$$

$$= 33.4 \text{ billion dollars.}$$

SOLUTIONS

3. The New England Aquarium is planning a new exhibit to house mantis shrimp. Mantis shrimp (*Squilla empusa*) are small crustaceans with very powerful claws. Tanks designed to hold mantis shrimp need to be specially constructed to withstand blows from the shrimps' powerful claws. (Several exhibits of mantis shrimp have been destroyed - by the shrimp in the exhibit - when the shrimp struck the bottom of the tank and broke the glass of the tank.)

The New England Aquarium are planning to build a tank that has glass sides and a slate bottom (see diagram below). The tank will have a square bottom, no top, and should have a volume of 100 cubic feet.



- (a) Slate costs \$10 per square foot and glass costs \$2 per square foot. Find a formula for the cost of the aquarium.

$$\text{Volume} = x^2 y = 100 \quad \text{so} \quad y = 100/x^2.$$

$$\begin{aligned} \text{Cost} &= 10x^2 + 8xy \quad \leftarrow \text{substitute } y = \frac{100}{x^2} \text{ here.} \\ &= 10x^2 + \frac{800}{x} \\ &= C(x). \end{aligned}$$

- (b) If the New England Aquarium wants to build the cheapest tank possible, what dimensions should the tank have?

$$C'(x) = 20x - \frac{800}{x^2} = 0 \quad \text{so} \quad x = (40)^{1/3}$$

$$C''(x) = 20 + \frac{1600}{x^3} \quad \text{so} \quad C''((40)^{1/3}) > 0 \quad \text{and}$$

the critical point is a local minimum.

$$\text{Dimensions of tank: } x = 40^{1/3} \text{ feet}$$

$$y = \frac{100}{40^{2/3}} \text{ feet.}$$

- (c) How do you know that the dimensions that you gave in part (b) will give the cheapest tank possible?

$$C''(40^{1/3}) > 0, \quad \text{as shown above.}$$

SOLUTIONS

4. In this problem, the function $f(x)$ will always refer to the function defined by the equation:

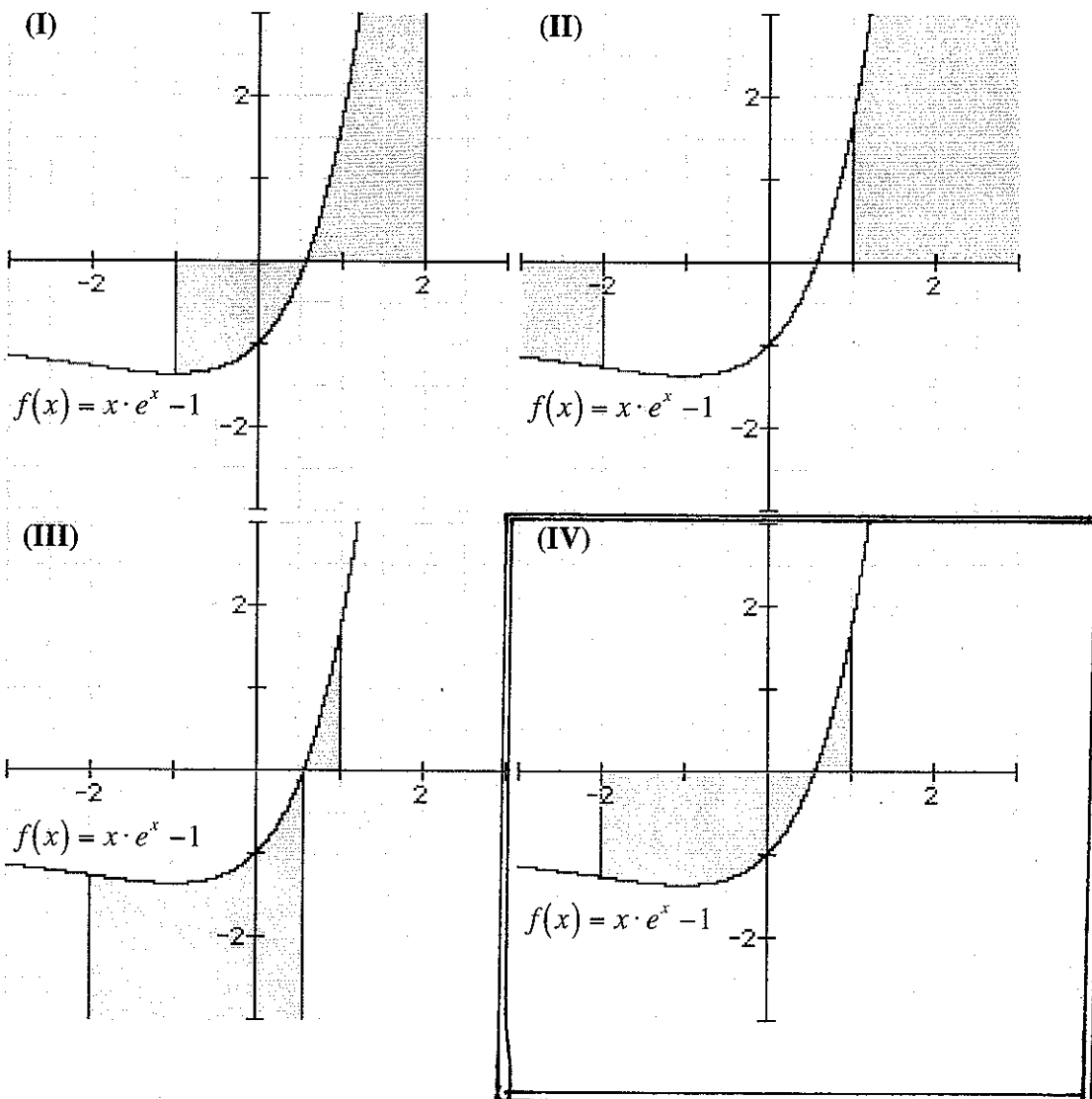
$$f(x) = x \cdot e^x - 1,$$

and the function $F(x)$ will always refer to the function defined by the equation:

$$F(x) = x \cdot e^x - e^x - x + 100.$$

- (a) Circle the shaded area (shown below) that most closely corresponds to the symbols:

$$\int_{-2}^1 f(x) \cdot dx.$$



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SOLUTIONS

- (b) Show that the function $F(x)$ is an anti-derivative of $f(x)$.

$$F(x) = x e^x - e^x - x + 100$$

$$F'(x) = e^x + x e^x - e^x - 1$$

$$= x e^x - 1$$

$$= f(x).$$

- (c) Find the numerical value of the integral:

$$\int_{-2}^1 f(x) \cdot dx.$$

$$\begin{aligned} \int_{-2}^1 f(x) dx &= F(1) - F(-2) \\ &= -2.59399415 \end{aligned}$$

- (d) The function $G(x)$ defined by the equation:

$$G(x) = x \cdot e^x - e^x - x + 100$$

is also an anti-derivative of $f(x)$. (**YOU ARE NOT REQUIRED TO SHOW THIS.**) If you were to use $G(x)$ to find the numerical value of the integral

$$\int_{-2}^1 f(x) \cdot dx$$

would you expect to get the same answer as you obtained in Part (c)? In a sentence or two briefly explain why or why not.

Yes, the name of the antiderivative is irrelevant.

SOLUTIONS

5. In this problem you are required to calculate formulas for each of the anti-derivatives (or indefinite integrals) listed below. Each of these problems can be solved using the technique of u-substitution, although you are welcome to use whatever methods you choose. Your answers should each include an unspecified constant along the lines of "+C."

(a) $\int \frac{10 \cdot (1 + \sqrt{x})^9}{\sqrt{x}} \cdot dx$

Let $u = 1 + \sqrt{x}$. $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$$\int \frac{10 (1 + \sqrt{x})^9}{\sqrt{x}} dx = \int \frac{10 u^9}{\sqrt{x}} 2\sqrt{x} du$$

$$dx = 2\sqrt{x} du$$

$$= 20 \int u^9 du$$

$$= 2u^{10} + C$$

$$= 2(1 + \sqrt{x})^{10} + C$$

(b) $\int \frac{e^x}{2 + e^x} \cdot dx$

Let $u = 2 + e^x$ $\frac{du}{dx} = e^x$

$$\int \frac{e^x}{2 + e^x} dx = \int \frac{e^x}{u} \cdot \frac{du}{e^x}$$

$$\frac{du}{e^x} = dx$$

$$= \int \frac{1}{u} du$$

$$= \ln(|u|) + C$$

$$= \ln(|2 + e^x|) + C$$

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SOLUTIONS

(c) $\int \frac{24 \cdot x^3 + 4}{(3 \cdot x^4 + 2 \cdot x + 1)^9} \cdot dx$

Let $u = 3x^4 + 2x + 1$

$$\frac{du}{dx} = 12x^3 + 2$$

$$dx = \frac{du}{12x^3 + 2}$$

$$\int \frac{24x^3 + 4}{(3x^4 + 2x + 1)^9} dx$$

$$= \int \frac{2}{u^9} du$$

$$= \int 2u^{-9} du = \frac{2}{-8} u^{-8} + C$$

$$= -\frac{1}{4} (3x^4 + 2x + 1)^{-8} + C$$

(d) $\int \frac{1}{x^2} \cdot e^{\frac{1}{x}} \cdot dx$

Let $u = \frac{1}{x}$. $\frac{du}{dx} = \frac{-1}{x^2}$

$$dx = -x^2 du$$

$$\int \frac{1}{x^2} e^{\frac{1}{x}} dx = -\int e^u du$$

$$= -e^u + C$$

$$= -e^{\frac{1}{x}} + C$$

SOLUTIONS

6. Although familiar to many in the United States, cable satellite and other forms of “pay television” are only gradually catching on in the United Kingdom (UK). However, British government forecasts suggest that by the year 2009, commercial television providers will reap annual revenues in excess of £11 billion.

Let T = the number of years since 1993. Let $V(T)$ = annual revenues from commercial television in the UK in units of billions of pounds (£). In 1993 ($T = 0$) annual revenues from commercial television services in the UK totaled approximately £4 billion.

- (a) The subsequent growth of revenues from commercial television in the UK is reasonably well represented by the differential equation:

$$V'(T) = -0.22 \cdot V(T) + 2.574.$$

Use this information and Euler's Method to complete the table shown below.

Time (T in years)	Commercial television revenue (£ billions)	Rate of change (in billions of £ per year)	Amount commercial television revenue will change in the next four years	New commercial television revenue (£ billions)
$T=0$	4	1.694	6.776	10.776
$T=4$	10.776	0.20328	0.81312	11.58912
$T=8$	11.58912	0.0243936	0.0975744	11.6866944
$T=12$	11.6866944	0.002927232	0.011708928	11.69840333
$T=16$	11.69840333			

- (b) Based on your calculations, do you think that the British government are justified when they predict that by the year 2009 ($T = 16$) annual revenues from commercial television in the UK will surpass £11 billion?

Yes, they are probably justified as our estimate for $V(16)$ exceeds £11 billion.

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SOLUTIONS

- (c) A formula for the function $V(T)$ is given below. Is this formula a symbolic solution for the differential equation in this problem? Does the formula satisfy the initial condition in this problem? Remember to provide evidence to justify your answers.

$$V(T) = 11.7 - 7.7 \cdot e^{-0.22T}$$

$$\text{LHS} = V'(T) = -7.7 e^{-0.22T} (-0.22)$$

$$\begin{aligned} \text{RHS} &= -0.22 (11.7 - 7.7 e^{-0.22T}) + 2.574 \\ &= -7.7 e^{-0.22T} (-0.22). \end{aligned}$$

LHS = RHS so this formula is a solution of the differential equation $V'(t) = -0.22 \cdot V(t) + 2.574$.

$$V(0) = 11.7 - 7.7 e^0 = 4.$$

The formula is also consistent with the initial condition.

- (d) Consider your answers to Questions (b) and (c). Based on your calculations, can you say with total confidence that the annual revenue in 2009 (i.e. $T = 16$) will be greater than £11 billion? Briefly explain why or why not.

Using the formula with $T = 16$ gives:

$$V(16) = 11.7 - 7.7 e^{-0.22(16)} = 11.47208435$$

The exact figure exceeds £11 billion, so we can say with total confidence that $V(16) > 11$.

SOLUTIONS

7. The Barry Manilow fan club of Chapel Hill plan to produce official Barry Manilow T-shirts, and sell them for \$7 each. Inquiries to local T-shirt manufacturers indicate that the cost of producing q T-shirts will be given by

$$C(q) = 0.01q^3 - 0.6q^2 + 13q.$$

- (a) Assuming that the fan club is somehow able to sell every Barry Manilow T-shirt that they manufacture at a price of \$7 per shirt, how many T-shirts should they produce if they want to maximize profits? What will be their maximum profit?

$$\text{Revenue} = 7q$$

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$= 7q - 0.01q^3 + 0.6q^2 - 13q$$

$$= -0.01q^3 + 0.6q^2 - 6q$$

$$= p(q).$$

To find the local maximums and minimums, solve $p'(q) = 0$ for q .

$$p'(q) = -0.03q^2 + 1.2q - 6 = 0.$$

Using the Quadratic formula:

$$q = \frac{-1.2 \pm \sqrt{(1.2)^2 - 4(-0.03)(-6)}}{-0.06} = 5.858, 34.1421$$

To check which is the maximum, use $p''(q) = -0.06q + 1.2$

q	$p''(q)$	Classification
5.858	+0.8485	Local minimum
34.14	-0.8485	Local maximum.

The club should produce 34 t-shirts to get a maximum profit of:
\$96.56

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SOLUTIONS

From now on, assume that the fan club has decided to manufacture exactly 34 T-shirts, and that their production costs are always given by $C(34)$. We are now dropping the assumption that the club manages to sell all of their T-shirts, i.e. they can now sell fewer than 34 T-shirts.

A club member reckons that they will sell all 34 shirts at a price of \$7, but that for every dollar they raise prices, they will sell 2 fewer shirts. That is, if they raise the price to \$8, they will only sell 32 shirts, if they raise the price to \$9, they will only sell 30 shirts, etc.

(b) Find a formula for the profits as a function of the price of the shirts.

Let $x = \#$ of one dollar price increases.

$$\text{Price} = 7 + x \quad \# \text{ sold} = 34 - 2x.$$

$$\text{Revenue} = (7 + x)(34 - 2x) = 238 + 20x - 2x^2$$

$$\text{Cost} = C(34) = 141.44.$$

$$\begin{aligned} \text{Profit} = p(x) &= 238 + 20x - 2x^2 - 141.44 \\ &= 96.56 + 20x - 2x^2 \end{aligned}$$

(c) What price should the fan club charge for official Barry Manilow T-shirts if they want to maximize profits?

$$p'(x) = 20 - 4x = 0$$

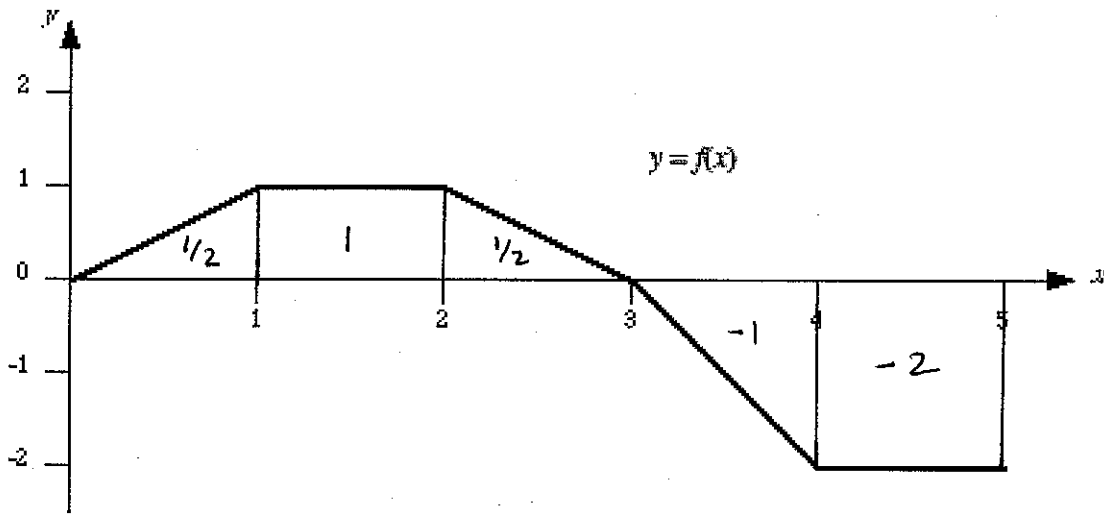
$$x = 5.$$

$$p''(x) = -4 < 0 \quad \text{so } x = 5 \text{ is a local maximum.}$$

The fan club should charge \$2 per t-shirt.

SOLUTIONS

8. In this problem you will be studying a function called $F(x)$. The derivative of $F(x)$ is called $f(x)$, i.e. $F'(x) = f(x)$. The graph below is a graph of $y = f(x)$.



Suppose that $F(5) = 7$. Use the information provided to answer the following questions.

- (a) What are the values of $F(1)$, $F(2)$, $F(3)$, and $F(4)$?

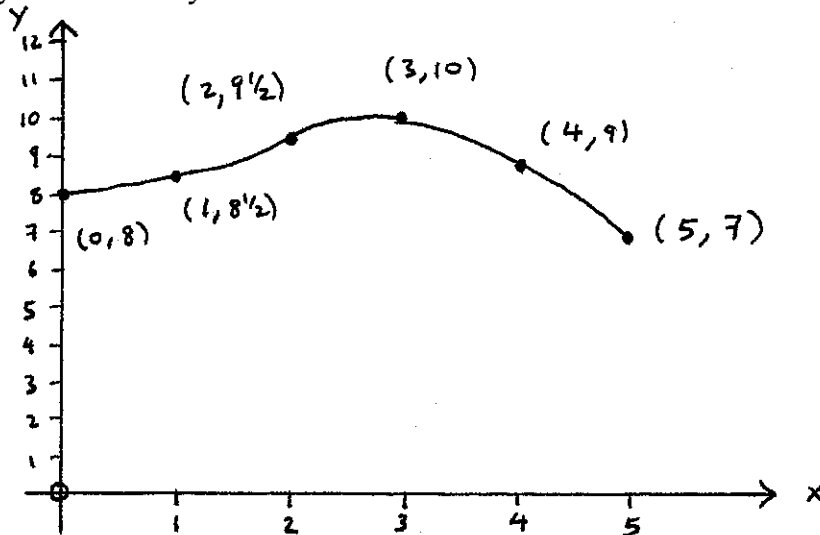
$$F(5) - F(4) = \int_4^5 f(x) dx = -2 \quad \text{so } F(4) = F(5) + 2 = 9.$$

$$F(4) - F(3) = \int_3^4 f(x) dx = -1 \quad \text{so } F(3) = F(4) + 1 = 10.$$

$$F(3) - F(2) = \int_2^3 f(x) dx = 1/2 \quad \text{so } F(2) = F(3) - 1/2 = 9\frac{1}{2}$$

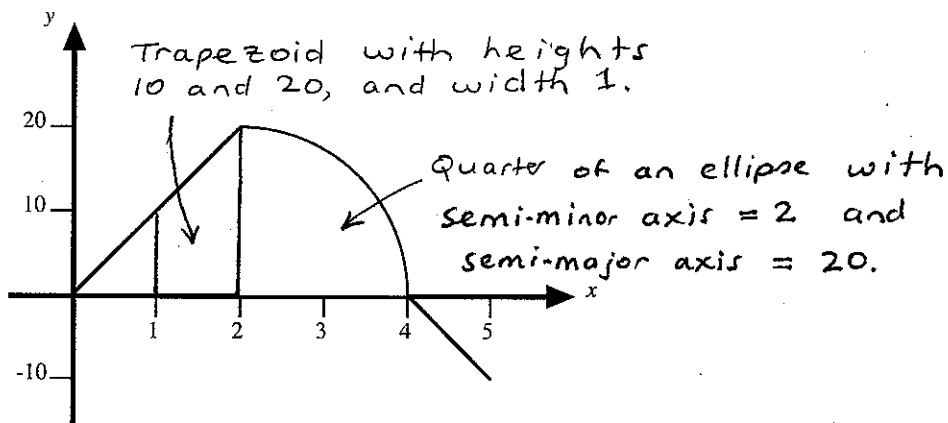
$$F(2) - F(1) = \int_1^2 f(x) dx = 1 \quad \text{so } F(1) = F(2) - 1 = 8\frac{1}{2}.$$

- (b) Sketch an accurate graph of $y = F(x)$. Be careful to label any points on the graph whose coordinates you know exactly.



SOLUTIONS

9. The derivative, F' , of a function F is shown on the graph below.



Suppose that $F(1) = 100$. What is the global maximum of F on the interval $[0, 5]$? At what value of x is this maximum achieved? Explain your reasoning.

The global maximum is achieved at $x = 4$.

This is because the function F has positive derivative on the interval $0 < x < 4$, so is increasing on this interval. After $x = 4$, the derivative of F becomes negative, so $x = 4$ is where F will have its maximum value.

$$\begin{aligned}
 F(4) - F(1) &= \int_1^4 F'(x) dx \\
 &= \frac{1}{2} (10+20) \cdot (1) + \frac{1}{4} \pi (2)(20) \\
 &= 15 + 10\pi
 \end{aligned}$$

Global maximum of F is $F(4) = F(1) + 15 + 10\pi$
 ≈ 146.4159265

SOLUTIONS.

10. Use Newton's method to solve each of the equations shown below. Your answers should be correct to eight (8) decimal places.

(a) $e^{-x} = 2 + x$ Write $f(x) = e^{-x} - 2 - x$ and
 $f'(x) = -e^{-x} - 1.$

Start with a guess for x_1 . Since $e^0 < 1$, this guess should be negative. Guess $x_1 = -0.5$

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
1	-0.5	0.1487212707	-2.648721271	-0.443851672
2	-0.443851672	0.0025509417	-2.55869927	-0.4428547038
3	-0.4428547038	7.7437376×10^{-7}	-2.557146071	-0.442854401
4	-0.442854401	0	-2.557145599	-0.442854401
5				

The solution is $x \approx -0.442854401.$

(b) $\ln(4-x^2) = x$ Write $f(x) = \ln(4-x^2) - x$ and
 $f'(x) = \frac{-2x}{4-x^2} - 1.$

There are two solutions to this equation, and depending on what value you choose for x_1 , you could get either one as your final answer.

Guess: $x_1 = -1.99$

Guess: $x_1 = 1.1$

Then Newton's method gives:

Then Newton's method gives:

$$x_2 = -1.97753026$$

$$x_2 = 1.05864851$$

$$x_3 = -1.96741777$$

$$x_3 = 1.05800655$$

$$x_4 = -1.96475281$$

$$x_4 = 1.05800640$$

$$x_5 = -1.96463580$$

$$x_5 = 1.05800640$$

$$x_6 = -1.96463560$$

The solutions are:

$$x_7 = -1.96463560$$

$$x = -1.96463560 \quad \text{and}$$

$$x = 1.05800640.$$