SOLUTIONS

Math 120 Winter 2009

Handout 17: In-Class Review for Exam 2

The topics covered by Exam 2 in the course include the following:

- Implicit differentiation.
- Finding formulas for tangent lines using implicit differentiation.
- Finding points on curves where the tangent line is horizontal or vertical.
- Related rates problems.
- Finding formulas for exponential functions.
- Solving equations that involve exponential functions using logarithms.
- Solving equations that involve logarithms.
- Interpreting the meaning of functions and inverses.
- The horizontal line test.
- Finding formulas for inverses of functions.
- Finding and using derivatives of inverse functions.
- Interpreting the meaning of the derivative of an inverse.
- Finding and using derivatives of functions that include exponential or logarithmic functions.
- Finding solutions to differential equations.
- Evaluating the constants in a differential equation.
- Using the solution of a differential equation to solve problems.
- Finding and using derivatives of functions that involve hyperbolic or inverse trigonometric functions.
- Using L'Hopital's rule to evaluate limits.

1. Let $P = f(t)$ give the US population in millions in the year t.

(a) What does the statement $f(2000) = 281$ tell you about the US population?

In the year 2000, the US population was 281 million people.

(b) Find and interpret the value of $f^{-1}(281)$. Give units with your answer.

$f^{-1}(281) = 2000$. The units are years. This means that the US population reached 281 million in the year 2000.

(c) What does the statement $f'(2000) = 3.476$ tell you about the US population? Give units with your answer.

$f'(2000) = 3.476$ has units of millions of people per year. This means that from the year 2000 to 2001, the US population will increase by approximately 3.476 million people.

(d) Evaluate and interpret the value of $\left(f^{-1}\right)'(281)$. Give units with your answer.

$(f^{-1})'(281) = \dfrac{1}{3.476}$ years per million people. This means that to go from 281 million people to 282 million people in the US, it will take approximately $1/3.476$ years.

SOLUTIONS

2. In this problem you will consider the values of x and y that satisfy the following equation:

$$x^2 + y^2 - 4x + 7y = 15.$$

(a) Calculate a formula for $\frac{dy}{dx}$.

$$2x + 2y \cdot \frac{dy}{dx} - 4 + 7 \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x + 4}{2y + 7}$$

(b) Find the equation of the tangent line to the point $(1, 2)$ on the curve.

Slope of tangent line $= m = \frac{2}{11}$

Equation: $y - 2 = \frac{2}{11}(x - 1)$

(c) Find the x and y coordinates of all points on the curve where the tangent line is horizontal.

The tangent line is horizontal when $dy/dx = 0$.

$dy/dx = 0$ when $-2x + 4 = 0$, or $x = 2$.

To find y, plug $x = 2$ into the equation:

$$(2)^2 + y^2 - 4(2) + 7y = 15$$

$$y = \frac{-7 \pm \sqrt{49 - 4(1)(-19)}}{2(1)} \approx -9.09, 2.09$$

The points are:
$(2, -9.09)$
$(2, 2.09)$

(d) Find the x and y coordinates of all points on the curve where the tangent line is vertical.

The tangent line is vertical when dy/dx is undefined. dy/dx is undefined when $2y + 7 = 0$ or $y = -7/2$. To find x, plug $y = -7/2$ into the above equation:

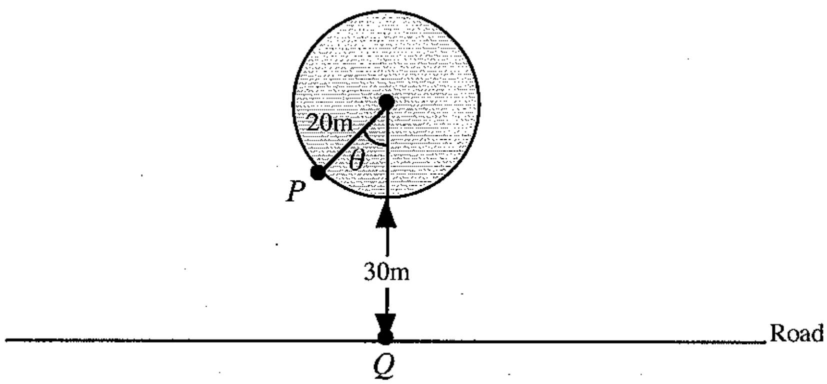
$$x^2 + (-7/2)^2 - 4x + 7(-7/2) = 15.$$

$$x = \frac{4 \pm \sqrt{16 - 4(1)(-109/4)}}{2(1)} \approx 7.59, -3.59$$

The points are $(-3.59, -7/2)$ and $(7.59, -7/2)$.

SOLUTIONS

3. A circular region is irrigated by a 20 meter long pipe, fixed at one end and rotating horizontally, spraying water. One rotation takes 5 minutes. A road passes 30 meters from the edge of the circular area (see diagram below).



(a) How fast is the end of the pipe, P, moving?

The pipe covers $2\pi(20) \approx 125.6637$ m in 5 minutes or 300 seconds. Speed of point P is:

$$\frac{40\pi}{300} \approx \frac{125.6637}{300} = 0.41888 \text{ m/s.}$$

(b) How fast is the distance PQ changing when $\theta = \pi/2$ radians?

As a function of θ, the distance between P and Q is given by:

$$D = \sqrt{(50 - 20\cos(\theta))^2 + (20\sin(\theta))^2}$$

$$D'(\theta) = \frac{2(50-20\cos(\theta))\cdot 20\sin(\theta) + 40\cdot\sin(\theta)\cdot\cos(\theta)}{2\sqrt{(50-20\cos(\theta))^2 + (20\sin(\theta))^2}}$$

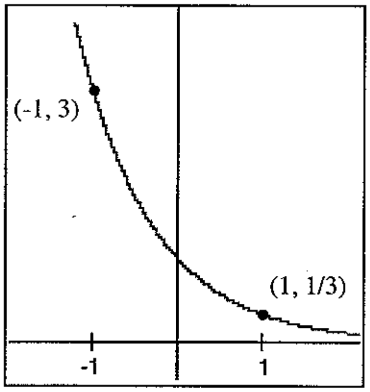
$D'(\pi/2) \approx 18.57$ m/radian. So, $\frac{dD}{dt} = D'(\pi/2)\cdot\frac{2\pi}{300} \approx 0.39$ m/s.

(c) How fast is the distance PQ changing when $\theta = 0$ radians?

$$D'(0) = 0 \quad \text{so} \quad \left.\frac{dD}{dt}\right|_{\theta=0} = 0 \text{ m/s}$$

SOLUTIONS

4. (a) Find the formula of the exponential function shown below.



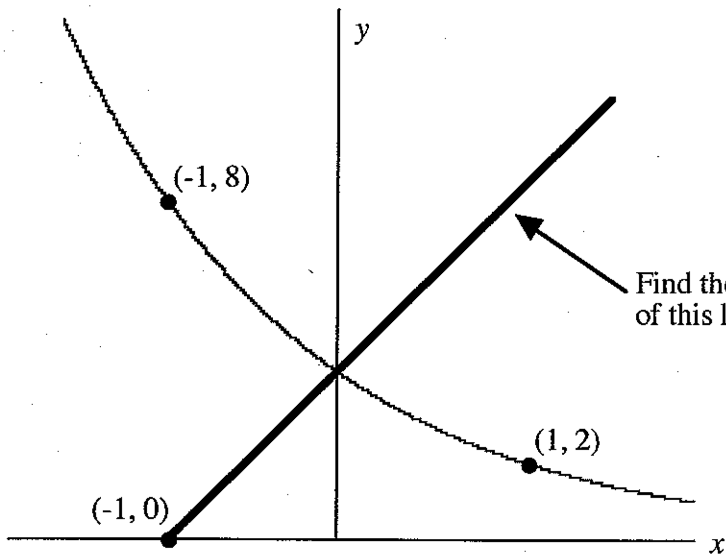
$$B = \left(\frac{1/3}{3}\right)^{\frac{1}{1--1}} = \frac{1}{3}$$

$$y = A \cdot \left(\frac{1}{3}\right)^x$$

$$\frac{1}{3} = A \cdot \left(\frac{1}{3}\right)^1 \quad \text{so} \quad A = 1.$$

$$y = 1 \cdot \left(\frac{1}{3}\right)^x$$

(b) Find the formula of the **LINEAR FUNCTION** shown below. Note that the curve shown is part of the graph of an **exponential** function.



Equation of exponential: $B = \left(\frac{2}{8}\right)^{\frac{1}{1--1}} = \frac{1}{2}$

$$y = A \cdot \left(\frac{1}{2}\right)^x$$

$$2 = A \cdot \left(\frac{1}{2}\right)^1 \quad \text{so} \quad A = 4.$$

Equation of linear function: $m = \frac{4 - 0}{0 - -1} = 4$

$$y = 4(x+1)$$

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5. Suppose that $x = \log(A)$ and that $y = \log(B)$ where A and B are both positive numbers. Write the following expressions in terms of x and y. (There should be no A or B in any of your final answers.) Use the laws of logarithms and exponentials to simplify your answers whenever possible.

(a) $\log(100AB)$.

$$= \log(100) + \log(A) + \log(B)$$

$$= 2 + x + y$$

(b) $\log\left(\dfrac{A^7}{\sqrt{B}}\right)$.

$$= \log(A^7) - \log(\sqrt{B})$$

$$= 7 \cdot \log(A) - \tfrac{1}{2}\log(B)$$

$$= 7x - \tfrac{1}{2}y$$

(c) $\log\left(A^B\right)$.

$$= B \cdot \log(A)$$

$$= B \cdot x$$

$$= 10^y \cdot x$$

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6. In this problem the letter x will always represent the number of years since 1990 and the letter y will always represent the number of people in the world who have a cell phone. **The units of y are millions of people.**

(a) The table given below shows some of the values of x and y. Based on the entries in the table, is y an exponential function of x? You must show your work here.

x	6	8	10	12
y	150	300	600	1200

$$B = \left(\frac{300}{150}\right)^{\frac{1}{8-6}} = \sqrt{2}$$

$$B = \left(\frac{1200}{600}\right)^{\frac{1}{12-10}} = \sqrt{2}$$

Get the same value of B no matter which points are used to compute it - exponential function will work well.

(b) In 1996 a total of 150 million people had cell phones. In 2000 a total of 600 million people had cell phones. Find a formula for exponential function that represents the relationship between x and y.

$$y = A\cdot(\sqrt{2})^x$$

$$150 = A\cdot(\sqrt{2})^6$$

$$A = \frac{150}{8} = 18.75$$

$$y = (18.75)\cdot(\sqrt{2})^6$$

(c) According to the formula that you found in Part (b), how many people will have cell phones in the year 2015?

$x = 25.$

$$y = (18.75)(\sqrt{2})^{25} \approx 108\,611.601\,600$$

In 2015, approximately 108,611,601,600 people will have cell phones.

Continued on the next page.

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(d) Use algebra and the formula that you found in Part (b), in what **year** will the number of people with cell phones reach 2.5 billion ($y = 2500$)?

$$2500 = 18.75\left(\sqrt{2}\right)^x$$

$$\frac{2500}{18.75} = \left(\sqrt{2}\right)^x$$

$$x = \frac{1}{\ln(\sqrt{2})} \ln\left(\frac{2500}{18.75}\right) \approx 14.11778738$$

The year will be 2004.

(e) **(6 points)** According to the CIA, the population of the entire world is given by the exponential function:

$$y = 5418 \cdot (1.0117)^x.$$

Based on this formula and the exponential formula that you found in Part (b), in what **year** would you expect every person in the world to have a cell phone?

Assuming a ratio of one phone to one person, this will occur when:

$$18.75\left(\sqrt{2}\right)^x = 5418\left(1.0117\right)^x$$

$$\left(\frac{\sqrt{2}}{1.0117}\right)^x = \frac{5418}{18.75}$$

$$x = \frac{1}{\ln\left(\frac{\sqrt{2}}{1.0117}\right)} \cdot \ln\left(\frac{5418}{18.75}\right)$$

$$\approx 16.917$$

This could happen in the year 2006 or 2007.

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7. Calculate each of the limits listed below.

(a) $\lim_{x \to 0} \frac{e^x - x - 1}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}.$

(b) $\lim_{x \to \infty} \frac{\ln(x)}{\sqrt[10]{x}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{10} x^{-0.9}} = \lim_{x \to \infty} \frac{10 x^{0.9}}{x} = 0.$

(c) $\lim_{u \to 0} \frac{u \cdot \tan^{-1}(u)}{1 - \cos(u)} = \lim_{u \to 0} \frac{\tan^{-1}(u) + \frac{u}{1+u^2}}{\sin(u)}$

$= \lim_{u \to 0} \frac{\frac{1}{1+u^2} + \frac{(1+u^2) - 2u^2}{(1+u^2)^2}}{\cos(u)}$

$= 2.$

(d) $\lim_{\theta \to 0} \frac{\sin(3\theta)}{\tan(5\theta)} = \lim_{\theta \to 0} \frac{3 \cdot \cos(3\theta)}{5 \cdot \sec^2(5\theta)} = \frac{3}{5}.$