Handout 14: Inverse Functions

The Concept of the Inverse

The mathematical relationship that reverses the roles of the input and output is called an **inverse relationship**. If a function uses x as its input and gives y as its output, then the **inverse relationship** takes y as the input and gives x as the output.

For example, if you read about the Uighurs and the Desert mummies in the handout on exponential functions, then you will have set up a function that takes the age of a desert mummy as the input and gives the amount of carbon-14 that remains in the mummy's tissues as the output.

INPUT	T	AGE IN YEARS
	↓	
FUNCTION	U	
	↓	
OUTPUT	$U(T)$	AMOUNT OF C-14

But what we were really interested in doing was starting with an amount of carbon-14 and determining the age of one of the desert mummies. To do this, we had to work back the other way – starting with the amount of carbon-14 that was present and going back to the age in years.

	T	AGE IN YEARS	$T = U^{-1}(c)$	
	↓		↑	
FUNCTION	U		U^{-1}	INVERSE
	↓		↑	
	$c = U(T)$	AMOUNT OF C-14	c	

The relationship that "undoes" or "reverses" the function is called the **inverse** of the function.

If $f(x)$ is a function and $g(x)$ is the inverse, then when you combine $f(x)$ and $g(x)$ by **composition**, the resulting composite function just gives x as the output.

$$f(g(x)) = x$$

$$g(f(x)) = x.$$

The inverse of the function $f(x)$ is often written using the symbols: $f^{-1}(x)$.

Calculating a Formula for the Inverse

When you have a formula for a function $f(x)$, you can sometimes find a formula for the inverse of the function.

Example

Find the equation for the inverse of:

$$y = f(x) = \frac{7 \cdot x}{3 \cdot x + 10}$$

Solution

Step 1: Write out the formula for the function, replacing *f*(*x*) by *y*

$$y = \frac{7 \cdot x}{3 \cdot x + 10}.$$

Step 2: Rearrange to make *x* the subject of the equation

$y \cdot (3 \cdot x + 10) = 7 \cdot x$	(Multiply both sides by $(3x + 10)$)
$3 \cdot x \cdot y + 10 \cdot y = 7 \cdot x$	(Distribute the y across the $(3x + 10)$)
$10 \cdot y = 7 \cdot x - 3 \cdot x \cdot y$	(Get everything involving x on the same side of the equation)
$10 \cdot y = (7 - 3 \cdot y) \cdot x$	(Factor out the x)
$\frac{10 \cdot y}{7 - 3 \cdot y} = x$	(Divide by $(7 - 3y)$)

The notation $f^{-1}(y) = \frac{10 \cdot y}{7 - 3 \cdot y}$ is sometimes used to denote the inverse of the function $f(x)$.

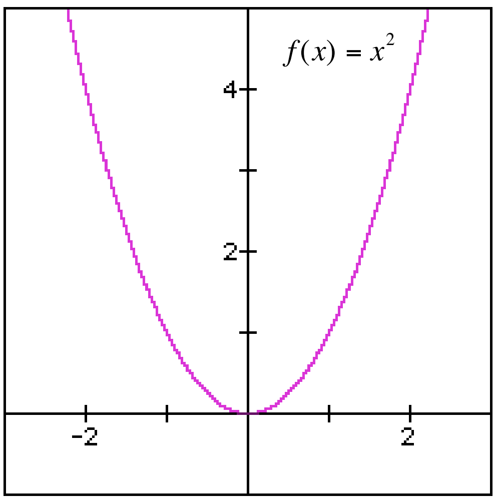
Is the inverse also a function? Some Inverses are Functions, Some are not

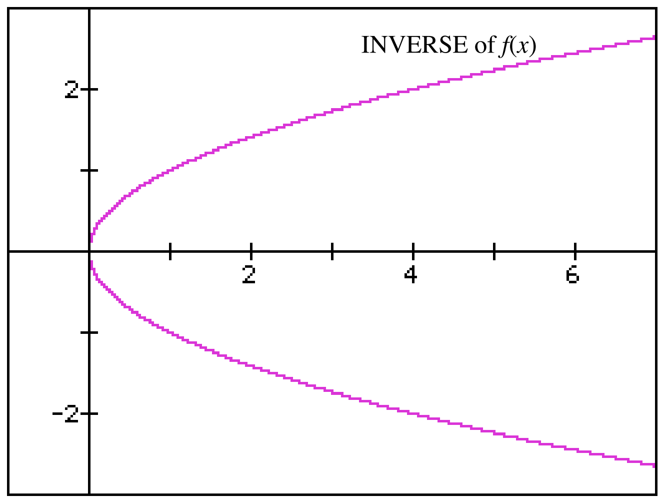
From the previous example, you might have been left with impression that it is always possible to come up with a formula for the inverse. It is sometimes possible to do this, but not always. The key is whether or not the **inverse is a function in its own right**.

Even some of the simplest functions do not have inverses that are functions in their own right. For example, the graph of the very simple quadratic function:

$$f(x) = x^2,$$

and its inverse are shown below. The graph of the inverse **fails** the vertical line test, and so this is not the graph of a function. Even when you are using a very simple function such as $f(x) = x^2$, the inverse may not be a function in its own right.





The question of whether or not the inverse is also a function in its own right depends a lot on the particular function $f(x)$ that you are talking about.

The test that can tell whether or not the inverse of a given $f(x)$ is a function in its own right or not is called the **Horizontal Line Test** and is described in the next section of these notes.

NOTE: If someone says that a function f is **invertible** (at least in Math 122) then **what that means is that the inverse of f is also a function in its own right.**

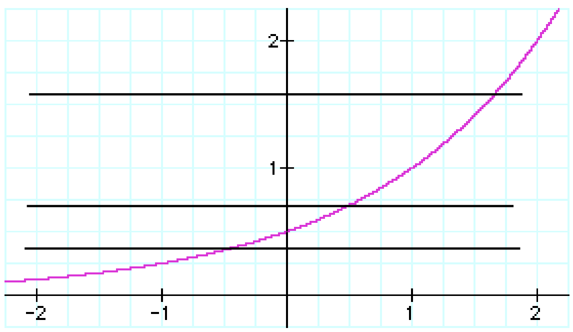
The Horizontal Line Test

This is always easiest to do when the function is represented in a graphical way. The test is known as the **Horizontal Line Test**, and it is essentially the vertical line test for functions except with x and y switched. (Hence why it uses horizontal rather than vertical lines.)

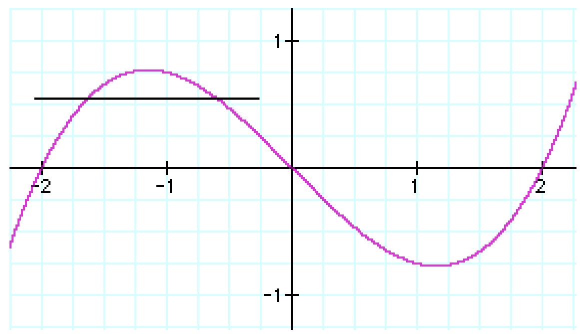
1. Draw the graph of the function in the normal way with the independent variable graphed on the horizontal axis and the dependent variable graphed on the vertical axis.

2. If every horizontal line that you can draw cuts the graph in **one** place (or misses the graph altogether) then the inverse is a function in its own right.

3. If any horizontal lines cut the graph in **more than one place**, the inverse is **not** a function in its own right.



Horizontal lines only cut in one place each. The inverse is a function in its own right.



Some horizontal lines only cut in more than one place. The inverse is not a function in its own right.

Sketching the Graph of the Inverse

When a function is represented in a graphical format, the inverse will be the graph obtained by reflecting in the line $y = x$.

The steps involved in doing this are as follows (see the next page for an illustration of this method).

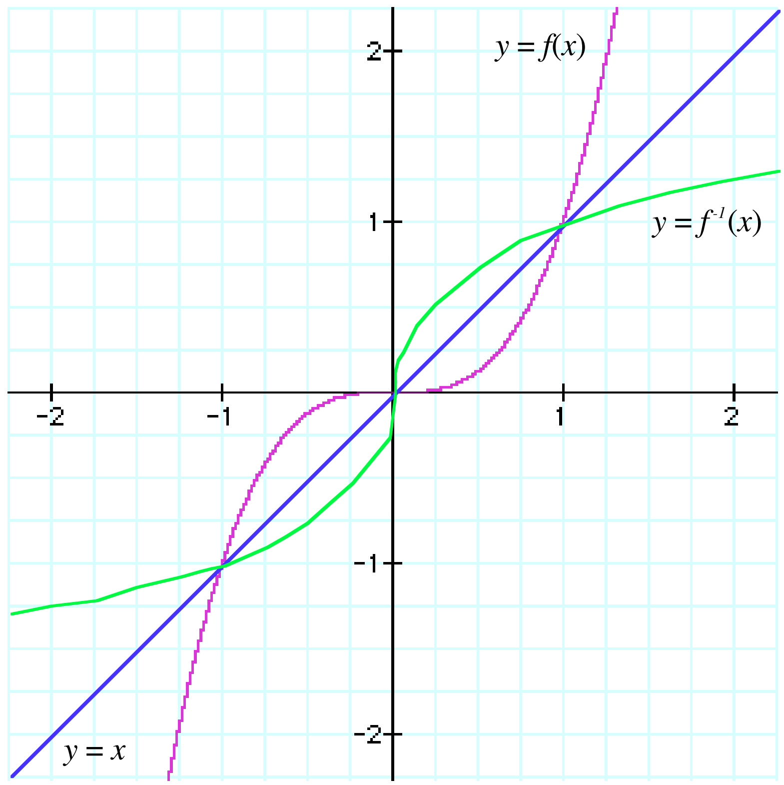
- **Step 1:** Using a well chosen set of axes, sketch the graph of the function, $f(x)$.

- **Step 2:** Draw in the diagonal line $y = x$. This is the 45° line that passes through the point (0, 0).

- **Step 3:** Turn your page 45° so that the line $y = x$ is vertical.

- **Step 4:** Using the vertical $y = x$ line as your mirror, draw in the mirror reflection of the graph of the function, $f(x)$.

- **Step 5:** Turn your page back to normal. The graph that you have just drawn by reflecting is the graph of the inverse.



In this diagram, the graph of the function is purple, the $y = x$ line is blue and the inverse graph is green. (Try turning the page 45º so that the blue line is vertical, and you will be able to see how the green graph is a reflection of the purple graph.)