Handout 13: Using the Properties of Logarithms to Solve Equations

The Laws of Logarithms

The logarithm function is very, very helpful when it comes to solving exponential equations like the one we solved for the age of the Desert mummies:

$$0.0000695 = (0.0001)\cdot(0.9998790392)^x$$

to find the age of the mummies, x. When working with exponential equations like the one shown above, it is usually easier to use logarithms to solve for x, rather than to use the GRAPH and INTERSECT capabilities of your calculator.

The important properties of the logarithm function are (A and B are assumed to be positive numbers):

a. $\log(A^T) = T\cdot\log(A)$
b. $\log(A\cdot B) = \log(A) + \log(B)$
c. $\log\left(\frac{A}{B}\right) = \log(A) - \log(B)$
d. $\log(10^T) = T$
e. $10^{\log(A)} = A$

Property (a) is the most important for the purposes of solving exponential equations like the one for the Desert mummies.

In this set of notes we will make use of Properties (b), (c) and (e) to solve equations that involve logarithms.

Overall Method for Solving Equations that Involve Logarithms

Equations that involve logarithms can often be solved using the following three steps:

1. Use either Property (b) or Property (c) to combine the logarithms into one, single logarithm.
2. Use Property (e) (sometimes called the Cobra-Mongoose rule) to eliminate the logarithm for the equation.
3. Use standard algebraic manipulations to solve for x.

Example

Solve the following equation to find x:

$$\log(x) - \log(3 - x) = 2.$$

Solution

First, you can use Property (c):

$$\log\left(\frac{A}{B}\right) = \log(A) - \log(B),$$

to combine the two logarithms into one logarithm. Doing this:

$$\log\left(\frac{x}{3 - x}\right) = 2.$$

Next, you can make **both sides** of this equation into the exponents of 10.

$$10^{\log\left(\frac{x}{3-x}\right)} = 10^2.$$

Now you can use Property (e) (the Cobra-Mongoose rule) to eliminate "log" from the equation:

$$\frac{x}{3 - x} = 10^2.$$

$$\frac{x}{3 - x} = 100.$$

(Note that $10^2 = 100$.) Now that the logarithm has been eliminated from the equation, all that remains is to perform algebraic manipulations to solve for x. Doing these:

$x = 100 \cdot (3 - x)$	(Multiply both sides by $(3 - x)$).
$x = 300 - 100 \cdot x$	(Distribute the 100 across $(3 - x)$).
$101 \cdot x = 300$	(Add $100 \cdot x$ to both sides of the equation).
$x = \dfrac{300}{101}.$	(Divide both sides of the equation by 101).

Example

Solve the following equation to find x:

$$\log(x) + \log(x + 21) = 2.$$

Solution

First, you can use Property (b):

$$\log(A \cdot B) = \log(A) + \log(B),$$

to combine the two logarithms into one logarithm. Doing this:

$$\log(x \cdot (x + 21)) = 2.$$

$$\log(x^2 + 21 \cdot x) = 2$$

Next, you can make **both sides** of this equation into the exponents of 10.

$$10^{\log(x^2 + 21 \cdot x)} = 10^2.$$

Now you can use Property (e) (the Cobra-Mongoose rule) to eliminate "log" from the equation:

$$x^2 + 21 \cdot x = 10^2.$$

$$x^2 + 21 \cdot x = 100.$$

(Note that $10^2 = 100$.) Now that the logarithm has been eliminated from the equation, all that remains is to perform algebraic manipulations to solve for x. Doing these:

$x^2 + 21 \cdot x - 100 = 0$ (Subtract 100 from both sides).

$(x + 25) \cdot (x - 4) = 0$ (Factor the quadratic).

From the factored form, the solutions appear to be $x = -25$ and $x = +4$. Of these two, only $x = +4$ is actually a solution of the equation:

$$\log(x) + \log(x + 21) = 2.$$

This is because the **domain of the logarithm function,** $g(x) = \log(x)$ consists only of **positive *x*-values**, i.e. $x > 0$. Therefore, only positive x-values are acceptable as solutions of this equation, and the only solution is $x = +4$.