MATH 120 – THIRD UNIT TEST

Wednesday, April 29, 2009.

NAME: SOLUTIONS

Circle the recitation section you attend

Tuesday, Thursday MORNING **A**

Tuesday, Thursday AFTERNOON **B**

Instructions:

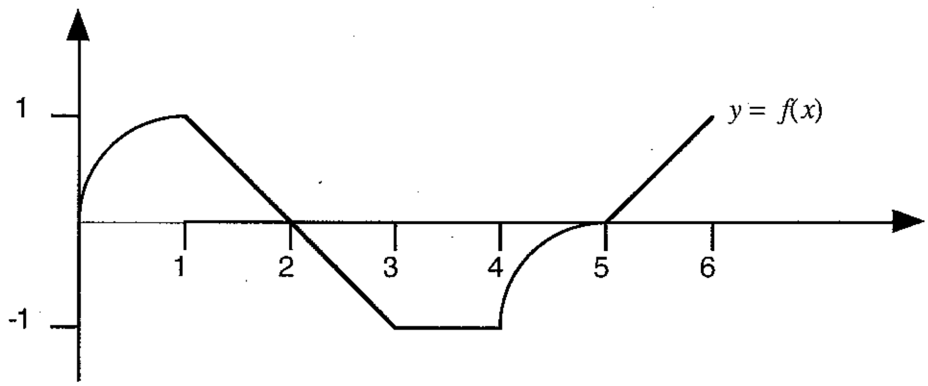
1. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
2. Please read the instructions for each individual question carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated. However, if you are asked to find an ***exact*** value of a quantity that involves an integral then you should not use calculator integration for this.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. Please **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

Problem	Total	Score
1	12	
2	24	
3	16	
4	10	
5	13	
6	25	
Total	100	

SOLUTIONS

1. 12 Points. CLEARLY INDICATE YOUR FINAL ANSWERS.

In this problem, the function $F(x)$ is defined by the formula $F'(x) = f(x)$, where $f(t)$ is the function graphed below and the value $F(5) = 6$.



You may assume that between $x = 0$ and $x = 1$ and also between $x = 4$ and $x = 5$, the shape of the graph forms one quarter of a circle with radius 1.

(a) **(8 points)** Find the following function values as accurately as possible:

$F(1) = F(2) - \int_1^2 f(x)dx = 8 - \frac{\pi}{4}$

$F(2) = F(3) - \int_2^3 f(x)dx = 8\frac{1}{2} - \frac{\pi}{4}$

$F(3) = F(4) - \int_3^4 f(x)dx = 8 - \pi/4$

$F(4) = F(5) - \int_4^5 f(x)dx = 7 - \frac{\pi}{4}$

(b) **(4 points)** Find the x coordinates of all critical points of the function $F(x)$ between $x = 0$ and $x = 6$. Classify each critical point as a local maximum, local minimum or neither. Record your answers in the table provided below.

x	Classification
2	Local maximum
5	Local minimum.

SOLUTIONS

2. 24 Points. SHOW ALL WORK. CLEARLY INDICATE YOUR ANSWERS.

In this problem you are required to calculate formulas for each of the anti-derivatives (or indefinite integrals) listed below. Each of these problems can be solved using the technique of u-substitution, although you are welcome to use whatever methods you choose. Your answers should each include an unspecified constant along the lines of "+*C*."

NOTE: You should not use your calculator to evaluate integrals in this problem, apart from working out arithmetic and evaluating functions.

(a) **(6 points)** $\int 7\cdot\left(x^2+x+1\right)^6\cdot[2x+1]\cdot dx = \int 7u^6\cdot[2x+1]\cdot\frac{du}{2x+1}$

$u = x^2+x+1$

$\frac{du}{dx} = 2x+1$

$dx = \frac{du}{2x+1}$

$= \int 7u^6\,du$

$= u^7 + C$

$= (x^2+x+1)^7 + C$

(b) **(6 points)** $\int \frac{1}{3}\cdot\left(x+\ln(x)\right)^{\frac{-2}{3}}\cdot\left[1+\frac{1}{x}\right]\cdot dx = \int \frac{1}{3}u^{-2/3}\cdot\left[1+\frac{1}{x}\right]\cdot\frac{du}{1+1/x}$

$u = x+\ln(x)$

$\frac{du}{dx} = 1+\frac{1}{x}$

$dx = \frac{du}{1+\frac{1}{x}}$

$= \int \frac{1}{3}u^{-2/3}\,du$

$= u^{1/3} + C$

$= (x+\ln(x))^{1/3} + C$

Continued on the next page.

SOLUTIONS

In this problem you are required to calculate formulas for each of the anti-derivatives (or indefinite integrals) listed below. Each of these problems can be solved using the technique of u-substitution, although you are welcome to use whatever methods you choose. Your answers should each include an unspecified constant along the lines of "+C."

NOTE: You should not use your calculator to evaluate integrals in this problem, apart from working out arithmetic and evaluating functions.

(c) **(6 points)** $\int \frac{9x^2+6}{x^3+2x+1} \cdot dx$ $= \int \frac{3(3x^2+2)}{u} \cdot \frac{du}{3x^2+2}$

$u = x^3 + 2x + 1$

$\frac{du}{dx} = 3x^2 + 2$

$dx = \frac{du}{3x^2+2}$

$= \int \frac{3}{u} \, du$

$= 3 \cdot \ln(|u|) + C$

$= 3 \cdot \ln(|x^3 + 2x + 1|) + C$

(d) **(6 points)** $\int \frac{1}{x} \cdot \frac{1}{\ln(x)} \cdot dx$ $= \int \frac{1}{x} \cdot \frac{1}{u} \cdot x \cdot du$

$u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x \cdot du$

$= \int \frac{1}{u} \, du$

$= \ln(|u|) + C$

$= \ln(|\ln(x)|) + C$

3. 16 Points. CLEARLY INDICATE YOUR ANSWERS.

In this problem the functions $f(x)$ and $F(x)$ will always refer to the functions defined by the formulas:

- $f(x) = x \cdot e^x$
- $F(x) = x \cdot e^x - e^x + 1.$

In this problem, $g(x)$ is also a function. All that you can assume about the function $g(x)$ is listed below.

- $g(2) = 3$
- $g(5) = 7$
- $g'(2) = 2$
- $g'(5) = -1$

(a) **(4 points)** Verify that $F(x)$ is an anti-derivative of $f(x)$. Be sure to show full details of your calculations.

$$F'(x) = e^x + x \cdot e^x - e^x$$

$$= x \cdot e^x$$

$$= f(x).$$

Since $F'(x) = f(x)$, $F(x)$ is an anti-derivative of $f(x)$.

(b) **(4 points)** Evaluate the numerical value of the definite integral: $\int_0^2 f(x) \cdot dx$.

$$\int_0^2 f(x) \cdot dx = F(2) - F(0)$$

$$= (2e^2 - e^2 + 1) - (0 \cdot e^0 - e^0 + 1)$$

$$= e^2 + 1$$

$$\approx 8.3891$$

Continued on the next page.

In this problem the functions $f(x)$ and $F(x)$ will always refer to the functions defined by the formulas:

- $f(x) = x \cdot e^x$
- $F(x) = x \cdot e^x - e^x + 1$.

In this problem, $g(x)$ is also a function. All that you can assume about the function $g(x)$ is listed below.

- $g(2) = 3$
- $g'(2) = 2$
- $g(5) = 7$
- $g'(5) = -1$

(c) **(4 points)** A new function $h(x)$ is defined by the equation:

$$h(x) = \frac{f'(x)}{g(x)} - \frac{f(x) \cdot g'(x)}{g(x)^2}.$$

Find a formula for the most general anti-derivative of $h(x)$. Your answer should contain one unspecified constant (i.e. $+C$).

$$h(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x)^2}.$$

This is the quotient rule.

Most general anti-derivative of $h(x)$ is:

$$\frac{f(x)}{g(x)} + C$$

(d) **(4 points)** Evaluate the exact numerical value of the definite integral: $\int_2^5 h(x) \cdot dx$.

$$\int_2^5 h(x) \cdot dx = \frac{f(5)}{g(5)} - \frac{f(2)}{g(2)}$$

$$= \frac{5e^5}{7} - \frac{2e^2}{3}$$

$$\approx 101.0834.$$

SOLUTIONS

4. **10 Points. CLEARLY INDICATE YOUR FINAL ANSWERS.**

(a) **(4 points)** Find the approximate value of $\int_0^4 (1.1)^{x^2}\,dx$ obtained when the integral is approximated using a left-hand Riemann sum and 100 rectangles. If you give your answer as a decimal, include at least eight (8) decimal places.

0 → A   4 → B   100 → N   (B − A) / N → W

(1.1) ^ (x²) → Y1.

sum(seq(Y1(A+K*W)*W, K, 0, N−1))

= 7.361529952

(b) **(4 points)** Find the approximate value of $\int_0^4 (1.1)^{x^2}\,dx$ obtained when the integral is approximated using a right-hand Riemann sum and 100 rectangles. If you give your answer as a decimal, include at least eight (8) decimal places.

sum(seq(Y1(A+K*W)*W, K, 1, N))

= 7.505328872

(c) **(2 points)** Find the best estimate of $\int_0^4 (1.1)^{x^2}\,dx$ obtained when the integral is approximated using 100 rectangles. If you give your answer as a decimal, include at least eight (8) decimal places.

$$\text{Best estimate} = \frac{\text{Left hand Sum} + \text{Right hand Sum}}{2}$$

= 7.43342941 2

5. 13 Points. NO WORK = NO CREDIT. CLEARLY INDICATE FINAL ANSWER.

Use **Newton's method** to find the solution of the equation:

$$\tfrac{1}{2}\cos(x) = x.$$

correct to five (5) decimal places. In order to get credit, you must show your work.

You should not use your calculator on this problem for anything except simple arithmetic, and evaluating trigonometric functions. In particular, you should not solve the equation by tracing the graph of a function or funding intersection points on your calculator or **using a program** to carry out Newton's method.

If you use your calculator, make sure it is set in RADIAN mode. Clearly indicate your final answer and include at least eight (8) decimal places.

We will write: $f(x) = \frac{1}{2}\cos(x) - x$ and

$$f'(x) = -\frac{1}{2}\sin(x) - 1.$$

Guess the value of x_1 to be $x_1 = 0.5$. Then according to Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Summarizing the iterative calculations in a table:

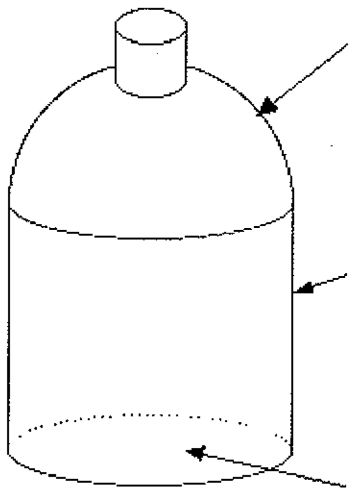
n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
1	0.5	−0.0612087191	−1.239712769	0.4506266931
2	0.4506266931	-5.39525248×10^{-4}	−1.217764876	0.4501836476
3	0.4501836476	-4.41768×10^{-8}	−1.217565446	0.4501836113

Accurate to 5 decimal places, the solution of $\frac{1}{2}\cos(x) = x$ is:

$$x \approx 0.4501836113$$

6. 25 Points. NO WORK = NO CREDIT. CLEARLY INDICATE YOUR ANSWERS.

The diagram given below shows the design of a water bottle. The bottle is made up of a cylinder and a hemisphere (half of a sphere).



This water bottle is supposed to hold 500 ml (500 cubic centimeters) of water. To keep production costs low the water company wants to use a little plastic as possible when making the bottles. Your job in this problem will be to calculate the size of water bottle that has the least amount of *surface area.*

(a) **(6 points)** Find an equation that gives the total *surface area* of the bottle. (Do not worry about the cap on the top but DO worry about the circle of plastic on the bottom of the bottle.)

Let S = surface area.

$$S = 2\pi r^2 + 2\pi r h + \pi r^2$$
$$= 3\pi r^2 + 2\pi r h$$

(b) **(4 points)** Find an equation that gives the total volume enclosed by the bottle as a function of the height, *h*, and radius, *r*, of the bottle. (Do not worry about the cap on the top.)

Let V = volume.

$$V = \frac{2}{3}\pi r^3 + \pi r^2 h$$

Continued on the next page.

(c) **(4 points)** Combine your answers to Parts (a) and (b) to create an equation for the *surface area* of the bottle that only involves the variable r and constants.

$$V = \frac{2}{3}\pi r^3 + \pi r^2 h = 500 \quad \text{so} \quad h = \frac{500 - \frac{2}{3}\pi r^3}{\pi r^2}$$

$$S = 3\pi r^2 + 2\pi r\left(\frac{500 - \frac{2}{3}\pi r^3}{\pi r^2}\right) = \frac{5}{3}\pi r^2 + \frac{1000}{r}$$

(d) **(7 points)** Determine the numerical value of r that will give a bottle with the least amount of *surface area*.

$$\frac{dS}{dr} = \frac{10}{3}\pi r - \frac{1000}{r^2} = 0 \quad \text{so that}$$

$$\frac{10\pi}{3} r = \frac{1000}{r^2}$$

$$r^3 = \frac{300}{\pi}$$

$$r = \left(\frac{300}{\pi}\right)^{1/3} \approx 4.5708 \text{ cm}$$

(e) **(4 points)** Using either the first or second derivative of *surface area*, confirm that the value of r that you calculated in Part (d) really does minimize the *surface area* of the bottle.

$$\frac{d^2S}{dr^2} = \frac{10\pi}{3} + \frac{2000}{r^3}.$$ Substituting $r = 4.5708$ into this gives $\frac{d^2S}{dr^2} \approx 31.4157 > 0$. As the second derivative is positive, $r = 4.5708$ corresponds to a minimum value of surface area.