MATH 120 – SECOND UNIT TEST

Friday, March 20, 2009.

NAME: SOLUTIONS

Circle the recitation section you attend

Tuesday, Thursday MORNING **A**

Tuesday, Thursday AFTERNOON **B**

Instructions:

1. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
2. Please read the instructions for each individual question carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated. However, if you are asked to find an ***exact*** value of a quantity that involves an integral then you should not use calculator integration for this.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. Please **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

Problem	Total	Score
1	13	
2	19	
3	14	
4	15	
5	15	
6	24	
Total	100	

SOLUTIONS

1. 13 Points. SHOW WORK AND CLEARLY INDICATE FINAL ANSWERS.

The Folium of Descartes is a curve with the equation:

$$x^3 + y^3 = 3 \cdot x \cdot y.$$

(a) **(8 points)** Find a formula for $\frac{dy}{dx}$ for the Folium of Descartes.

Use implicit differentiation:

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3y + 3x \cdot \frac{dy}{dx}$$

$$3y^2 \cdot \frac{dy}{dx} - 3x \cdot \frac{dy}{dx} = 3y - 3x^2$$

$$\frac{dy}{dx} \cdot (3y^2 - 3x) = 3y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}.$$

(b) **(5 points)** Find the equation of the tangent line to the Folium of Descartes at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Slope of tangent line $= \dfrac{3(3/2) - 3(3/2)^2}{3(3/2)^2 - 3(3/2)} = -1.$

Equation of tangent line: $y - 3/2 = -1 \cdot (x - 3/2).$

SOLUTIONS

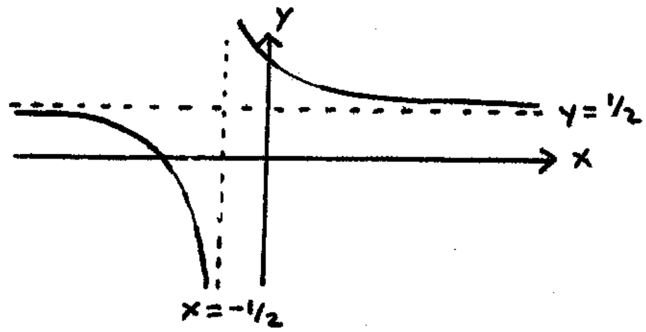
2. 19 Points. SHOW ALL WORK. CLEARLY INDICATE YOUR ANSWERS.

In this problem, the function $f(x)$ will always refer to:

$$f(x) = \frac{x+1}{2x+1}.$$

(a) **(7 points)** Find a formula for $f^{-1}(y)$. As part of your answer, you should demonstrate that such a formula exists.

Graph of $y = \dfrac{x+1}{2x+1}$:



This graph passes the horizontal line test.

$$y = \frac{x+1}{2x+1}$$

$$y(2x+1) = x+1$$

$$2xy + y = x + 1$$

$$2xy - x = 1 - y$$

$$x(2y-1) = 1 - y$$

$$x = \frac{1-y}{2y-1}$$

or

$$f^{-1}(y) = \frac{1-y}{2y-1}.$$

(b) **(3 points)** Evaluate $f^{-1}\left(\frac{2}{3}\right)$.

$$f^{-1}(2/3) = \frac{1 - 2/3}{2(2/3) - 1} = 1.$$

Continued on the next page.

SOLUTIONS

SHOW ALL WORK. CLEARLY INDICATE YOUR ANSWERS.

In this problem, the function $f(x)$ will always refer to:

$$f(x) = \frac{x+1}{2x+1}.$$

(c) **(4 points)** Evaluate $f'(1)$.

$$f'(x) = \frac{(2x+1) - 2(x+1)}{(2x+1)^2} = \frac{-1}{(2x+1)^2}$$

$$f'(1) = \frac{-1}{(2+1)^2} = \frac{-1}{9}.$$

(d) **(5 points)** Write down an equation for the tangent line to the curve $s = f^{-1}(t)$ at the point where $t = \frac{2}{3}$.

The slope of the tangent line is:

$$m = \frac{1}{f'(f^{-1}(2/3))} = \frac{1}{f'(1)} = -9.$$

The equation of the tangent line to the point $(t, s) = (2/3, 1)$ is:

$$s - 1 = -9(t - 2/3)$$

SOLUTIONS

3. 14 Points. SHOW YOUR WORK. NO WORK = NO CREDIT.

Carbon-14 is a naturally occurring radioactive element with a half life of 5730 years. This means that if you have 100 micrograms (μg) of carbon-14 today, after 5730 years, you will have only 50μg of carbon-14 left.

(a) **(3 points)** Let $M(t)$ represent the mass (in μg) of carbon-14 present in a piece of wood. $M(t)$ can be defined by a differential equation:

$$M'(t) = k \cdot M(t).$$

Write down a formula for the function $M(t)$. Your formula may contain up to two unspecified constants.

$$M(t) = \underbrace{A}_{\text{constant}} \cdot e^{\overbrace{k}^{\text{constant}} \cdot t}$$

(b) **(4 points)** The half life of carbon-14 is 5730 years. Use this to evaluate one of the constants in the formula that you found in Part (a).

When $t = 5730$, $M = A/2$.

$$\frac{A}{2} = A \cdot e^{k(5730)}$$

$$\frac{1}{2} = e^{k(5730)}$$

$$k = \frac{\ln(1/2)}{5730} \approx -0.00012097$$

(c) **(7 points)** A piece of living wood normally contains 0.001 μg of carbon-14. When the wood dies, the carbon-14 begins to decay. A piece of wood found buried at Stonehenge in England contained only 0.00063 μg of carbon-14. How old is this piece of wood? Show your work – no work = no credit.

$A = 0.001$. $T =$ age of buried wood.

$$0.00063 = 0.001 \cdot e^{-0.00012097\,T}$$

$$\frac{0.00063}{0.001} = e^{-0.00012097\,T}$$

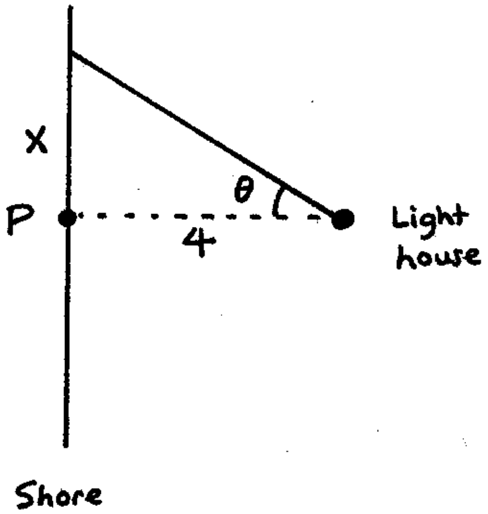
$$T = \frac{-1}{0.00012097} \ln\left(\frac{0.00063}{0.001}\right)$$

$$\approx 3819.482 \text{ years old.}$$

4. 15 Points. SHOW ALL WORK. CLEARLY INDICATE YOUR FINAL ANSWER.

A lighthouse is located on a small island, four (4) miles from the nearest point *P* on a straight shoreline. The light in the lighthouse makes two (2) revolutions per minute. How fast is the beam of light moving along the shoreline when it is half a mile from *P*?

Show your work (no work = no credit) and clearly write your final answer in the space provided below. Include appropriate units with your answer.



We want $\frac{dx}{dt}$ when $x = 1/2$.

Have: $\frac{d\theta}{dt} = 4\pi$ radians/minute.

We will begin by finding a relationship between x and θ.

$$\frac{x}{4} = \tan(\theta).$$

$$x = 4 \cdot \tan(\theta).$$

Now, $\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt} = 4 \cdot \sec^2(\theta) \cdot 4\pi$

When $x = 1/2$, $\sec(\theta) = \frac{\sqrt{4^2 + (1/2)^2}}{4}$ so:

$$\frac{dx}{dt} = 4 \cdot \left(\frac{\sqrt{4^2 + (1/2)^2}}{4}\right)^2 \cdot 4\pi \approx 51.0509$$

SPEED OF BEAM OF LIGHT ALONG SHORE: 51.0509 miles/minute

5. 15 Points. NO WORK = NO CREDIT. CLEARLY INDICATE FINAL ANSWERS.

Solve each of the following equations to find all possible solutions, x. You may use the following formula without justifying it:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)} \qquad a > 0.$$

You should not use your calculator on this problem for anything except simple arithmetic, evaluating logarithms and exponentials. In particular, you should not solve these equations by tracing the graph of a function or funding intersection points on your calculator.

(a) **(9 points)** $\log(x+1) - \log(x) = 2.$

$$\log\left(\frac{x+1}{x}\right) = 2$$

$$10^{\log\left(\frac{x+1}{x}\right)} = 10^2 = 100$$

$$\frac{x+1}{x} = 100$$

$$x + 1 = 100x$$

$$1 = 99x$$

Solution: $x = \frac{1}{99}.$

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NO WORK = NO CREDIT. CLEARLY INDICATE FINAL ANSWERS.

Solve each of the following equations to find all possible solutions, x. You may use the following formula without justifying it:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)} \qquad a > 0.$$

You should not use your calculator on this problem for anything except simple arithmetic, evaluating logarithms and exponentials. In particular, you should not solve these equations by tracing the graph of a function or funding intersection points on your calculator.

(b) **(6 points)** $\log(x) = \ln(x)$.

Note that $\log(x) = \log_{10}(x) = \dfrac{\ln(x)}{\ln(10)}$.

So, $\log(x) = \ln(x)$ is equivalent to:

$$\frac{\ln(x)}{\ln(10)} = \ln(x).$$

As $\ln(10) \neq 1$, the only solution of this equation is:

$$\ln(x) = 0$$

$$x = e^{\ln(x)} = e^0 = 1.$$

6. **24 Points. NO WORK = NO CREDIT. CLEARLY INDICATE FINAL ANSWERS.**

In each case, determine whether the limit exists. If the limit exists, find its value.

(a) **(8 points)** $\lim_{x \to 0} \frac{1-\cosh(5x)}{x^2}$.

As $x \to 0$, $\cosh(5x) \to 0$ so this is an indeterminant form $\left(\frac{0}{0}\right)$ and L'Hôpital's rule can be used.

$$\lim_{x \to 0} \frac{1 - \cosh(5x)}{x^2} = \lim_{x \to 0} \frac{-5\sinh(5x)}{2x}$$

This is also an indeterminant form $\left(\frac{0}{0}\right)$ so L'Hôpital's rule can be used again.

$$\lim_{x \to 0} \frac{1 - \cosh(5x)}{x^2} = \lim_{x \to 0} \frac{-25\cosh(5x)}{2}$$

$$= \frac{-25}{2}.$$

So, the limit exists and equals $-25/2$.

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NO WORK = NO CREDIT. CLEARLY INDICATE FINAL ANSWERS.

In each case, determine whether the limit exists. If the limit exists, find its value.

(b) **(8 points)** $\lim\limits_{x \to \infty} \dfrac{\ln(\ln(x))}{x \cdot \ln(x)}$.

As $x \to \infty$, this is an indeterminant form $\left(\frac{\infty}{\infty}\right)$ so L'Hôpital's rule can be used.

$$\lim_{x \to \infty} \frac{\ln(\ln(x))}{x \cdot \ln(x)} = \lim_{x \to \infty} \frac{\frac{1}{\ln(x)} \cdot \frac{1}{x}}{\ln(x) + 1}$$

$$= 0.$$

So, the limit exists and is equal to zero.

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NO WORK = NO CREDIT. CLEARLY INDICATE FINAL ANSWERS.

In each case, determine whether the limit exists. If the limit exists, find its value.

(c) **(8 points)** $\lim_{x \to 1}\left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right)$.

Note that when $x \neq 1$,

$$\frac{1}{\ln(x)} - \frac{1}{x-1} = \frac{x-1-\ln(x)}{(x-1)\cdot\ln(x)}$$

As $x \to 1$, this an indeterminant form $\left(\frac{0}{0}\right)$ so L'Hôpital's rule can be used.

$$\lim_{x \to 1}\left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right) = \lim_{x \to 1} \frac{1 - \frac{1}{x}}{\ln(x) + \frac{x-1}{x}}$$

This is still an indeterminant form $\left(\frac{0}{0}\right)$ so we can use L'Hôpital's rule again.

$$\lim_{x \to 1}\left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right) = \lim_{x \to 1} \frac{\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}.$$

So, the limit exists and is equal to $1/2$.