MATH 120 – SECOND UNIT TEST

Wednesday, April 1, 2009.

NAME: SOLUTIONS

Circle the recitation section you attend	Tuesday, Thursday MORNING A	Tuesday, Thursday AFTERNOON B

Instructions:

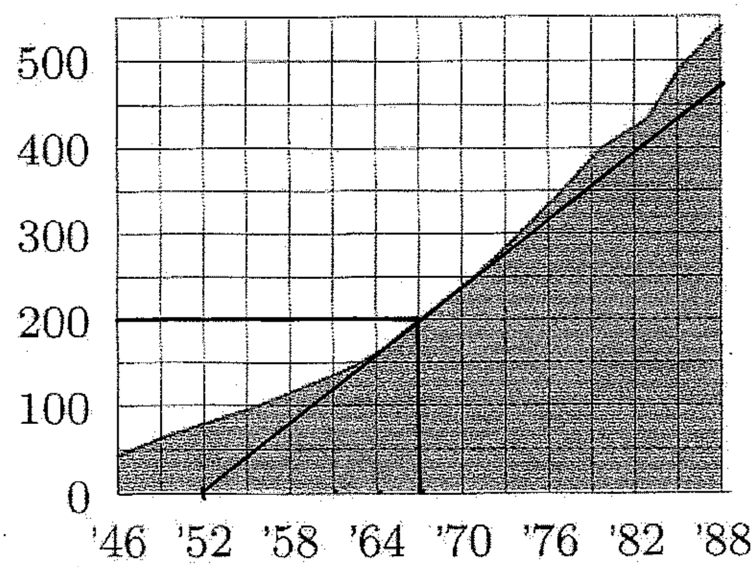
1. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
2. Please read the instructions for each individual question carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work for each exam question so that graders can see your final answer **and** how you obtained it.
4. You may use your calculator on all exam questions except where otherwise indicated. However, if you are asked to find an ***exact*** value of a quantity that involves an integral then you should not use calculator integration for this.
5. If you use graphs or tables to obtain an answer (especially if you create the graphs or tables on your calculator), be certain to provide an explanation and a sketch of the graph to show how you obtained your answer.
6. Please **TURN OFF** all cell phones and pagers, and **REMOVE** all headphones.

Problem	Total	Score
1	15	
2	14	
3	24	
4	15	
5	16	
6	16	
Total	100	

SOLUTIONS

1. 15 Points. SHOW WORK AND CLEARLY INDICATE FINAL ANSWERS.

The graph given below shows the number of motor vehicles, $f(t)$, in millions, registered in the world t years after 1946. Use this graph to estimate the values of each of the expressions listed below. Be sure to include **appropriate units** with each of your answers.



(a) **(3 points)** $f(21)$

$1946 + 21 = 1967$

$f(21) = 200$ million vehicles.

(b) **(3 points)** $f'(21)$

$f'(21) \approx \dfrac{200-0}{21-6} = 13\tfrac{1}{3}$ million vehicles per year.

(c) **(4 points)** $f^{-1}(200)$

$f^{-1}(200) = 21$ years since 1946.

(d) **(5 points)** $\left(f^{-1}\right)'(200)$

$(f^{-1})'(200) = \dfrac{1}{13\tfrac{1}{3}} = 0.075$

years per million vehicles.

SOLUTIONS

2. 14 Points. SHOW ALL WORK. CLEARLY INDICATE YOUR ANSWERS.

Consider the following equation that defines the relationship between the variables x and y.

$$x^3 + 2xy + y^2 = 4.$$

(a) **(8 points)** Find a formula for the derivative $\frac{dy}{dx}$.

$$x^3 + 2xy + y^2 = 4$$

$$3x^2 + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x\frac{dy}{dx} + 2y\frac{dy}{dx} = -3x^2 - 2y$$

$$(2x + 2y)\frac{dy}{dx} = -3x^2 - 2y$$

$$\frac{dy}{dx} = \frac{-3x^2 - 2y}{2x + 2y}$$

(b) **(6 points)** Find the equation of the tangent line to the curve $x^3 + 2xy + y^2 = 4$ at the point (1, 1).

Slope of tangent line $= \frac{-3-2}{2+2} = \frac{-5}{4}$.

Equation of tangent line:

$$y - 1 = \frac{-5}{4}(x - 1).$$

3. 24 Points. SHOW YOUR WORK.

The normal body temperature for a human being is 98.6°F. When a person dies, their body starts to cool down. The rate at which the body cools down is governed by Newton's Law of Cooling.

Newton's Law of Cooling states that the rate at which a body cools is proportional to the difference between the temperature of the body and the temperature of the environment.

Throughout this problem, time t will be measured in hours.

(a) **(6 points)** At time $t = 0$ an otherwise healthy normal person dies. The room that they were in is maintained at a steady temperature of 68°F. Find a formula of $H(t)$, the temperature of the body t hours after death. Your formula may contain one unspecified constant.

$$H(t) = 68 + 30.6\, e^{-k \cdot t}$$

(b) **(6 points)** During the first hour after death, the body cools by 2°F. Use this information to find the value of the unspecified constant in your answer to Part (a). If you give your answer as a decimal, include at least four (4) decimal places.

$$H(1) = 96.6$$

$$96.6 = 68 + 30.6\, e^{-k}$$

$$\frac{28.6}{30.6} = e^{-k}$$

$$k = -\ln\left(\frac{28.6}{30.6}\right) = 0.06759329$$

Continued on the next page.

The normal body temperature for a human being is 98.6°F. When a person dies, their body starts to cool down. The rate at which the body cools down is governed by Newton's Law of Cooling.

Newton's Law of Cooling states that the rate at which a body cools is proportional to the difference between the temperature of the body and the temperature of the environment.

Throughout this problem, time t will be measured in hours.

(c) **(6 points)** How many hours does it take for the rate at which the body is cooling to reach 1°F per hour? If you give your answer as a decimal, include at least four (4) decimal places.

$$H'(t) = (-0.06759329)(30.6)e^{-0.06759329t}$$

Set $H'(t) = -1$ and solve for t.

$$-1 = (-0.06759329)(30.6)e^{-0.06759329t}$$

$$\frac{-1}{(-0.06759329)(30.6)} = e^{-0.06759329t}$$

$$t = \frac{1}{(-0.06759329)} \cdot \ln\left(\frac{-1}{(-0.06759329)(30.6)}\right) \doteqdot 10.7518 \text{ hours.}$$

(d) **(6 points)** Suppose that when the body was found, the coroner measured the temperature of the body and found it to be 70°F. How many hours elapsed between the person dying and the coroner measuring the temperature? If you give your answer as a decimal, include at least four (4) decimal places.

$$70 = 68 + 30.6\, e^{-0.06759329t}$$

$$\frac{2}{30.6} = e^{-0.06759329t}$$

$$t = \frac{1}{-0.06759329} \cdot \ln\left(\frac{2}{30.6}\right)$$

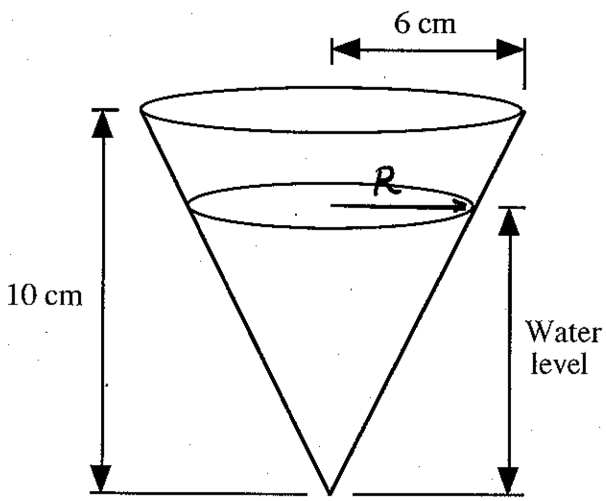
$$\approx 40.3568 \text{ hours.}$$

SOLUTIONS

4. 15 Points. NO WORK = NO CREDIT. CLEARLY INDICATE FINAL ANSWER.

A cone-shaped coffee filter has a radius of 6 cm and a height of 10 cm. Initially the filter is full of water, but immediately the water start leaks out of hole in the bottom of the filter at a rate of 1.5 cubic centimeters per second.

How fast is the water level in the filter falling when the water level is 8cm? Include appropriate units and write your final answer in the space provided below.



Let W = water level (in cm).

Want: $\frac{dW}{dt}$ when $W = 8$.

Have: $\frac{dV}{dt} = -1.5 \ cm^3/s$

Let R = radius at top of water (in cm).

Then: $V = \frac{1}{3} \pi \cdot R^2 \cdot W$.

Note that, by the principle of similar triangles:

$$\frac{R}{W} = \frac{6}{10} \quad \text{so} \quad R = \frac{6}{10} W.$$

This gives: $V = \frac{1}{3} \pi \left(\frac{6}{10} W\right)^2 \cdot W = \frac{36}{300} \pi \cdot W^3$.

Taking derivatives of both sides :

$$\frac{dV}{dt} = \frac{36}{300} \pi \cdot 3 \cdot W^2 \cdot \frac{dW}{dt} \quad \text{so :} \quad \frac{dW}{dt} = \frac{\frac{dV}{dt}}{\frac{36}{300} \pi \cdot 3 \cdot W^2}$$

Plugging $W = 8$ and $dV/dt = -1.5$ gives:

FINAL ANSWER: $dW/dt = -0.0207232 \ cm/s.$

5. **16 Points. NO WORK = NO CREDIT. CLEARLY INDICATE FINAL ANSWERS.**

Solve each of the following equations to find all possible solutions, x.

You should not use your calculator on this problem for anything except simple arithmetic, evaluating logarithms and exponentials. In particular, you should not solve these equations by tracing the graph of a function or funding intersection points on your calculator.

(a) **(8 points)** $3\cdot 5^x = 9\cdot 2^x$.

$$3\cdot 5^x = 9\cdot 2^x$$

$$5^x = 3\cdot 2^x$$

$$\ln(5^x) = \ln(3\cdot 2^x)$$

$$x\cdot \ln(5) = \ln(3) + x\cdot\ln(2)$$

$$x\cdot(\ln(5) - \ln(2)) = \ln(3)$$

$$x = \frac{\ln(3)}{\ln(5) - \ln(2)}$$

$$\approx 1.19897847$$

Continued on the next page.

Solve each of the following equations to find all possible solutions, x.

You should not use your calculator on this problem for anything except simple arithmetic, evaluating logarithms and exponentials. In particular, you should not solve these equations by tracing the graph of a function or funding intersection points on your calculator.

(b) **(8 points)** $\log(x) + \log(x+21) = 2.$

$$\log(x) + \log(x+21) = 2$$

$$\log(x \cdot (x+21)) = 2$$

$$10^{\log(x \cdot (x+21))} = 10^2$$

$$x \cdot (x+21) = 100$$

$$x^2 + 21x - 100 = 0.$$

Solving this equation using the quadratic formula gives:

$$x = \frac{-21 \pm \sqrt{21^2 - 4(1)(-100)}}{2}$$

$$= 4, \ -25.$$

Of these only $x = 4$ is a solution of $\log(x) + \log(x+21) = 2$ as the domain of $\log(x)$ includes only $x > 0$.

6. **16 Points. NO WORK = NO CREDIT. CLEARLY INDICATE FINAL ANSWERS.**

In each case, determine whether the limit exists. If the limit exists, find its value.

(a) **(8 points)** $\displaystyle \lim_{x \to 0} \frac{\tan(\pi \cdot x)}{\ln(1+x)}$.

As $x \to 0$, this is an indeterminant form of the $0/0$ type. We can apply L'Hôpital's rule to find the limit.

$$\lim_{x \to 0} \frac{\tan(\pi x)}{\ln(1+x)} = \lim_{x \to 0} \frac{\pi \cdot \sec^2(\pi x)}{\frac{1}{1+x}}$$

$$= \frac{\pi \cdot (1)^2}{1}$$

$$= \pi.$$

Continued on the next page.

NO WORK = NO CREDIT. CLEARLY INDICATE FINAL ANSWERS.

In each case, determine whether the limit exists. If the limit exists, find its value.

(b) **(8 points)** $\lim_{x \to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln(x)} \right).$

Before computing any limits, note that:

$$\frac{x}{x-1} - \frac{1}{\ln(x)} = \frac{x \cdot \ln(x) - x + 1}{(x-1) \cdot \ln(x)}.$$

As $x \to 1^+$, this is an indeterminant form of the $0/0$ type so we can apply L'Hôpital's rule.

$$\lim_{x \to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln(x)} \right) = \lim_{x \to 1^+} \frac{\ln(x) + 1 - 1}{\ln(x) + 1 - \frac{1}{x}}$$

This is also an indeterminant form of the $0/0$ type so we can apply L'Hôpital's rule again.

$$\lim_{x \to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln(x)} \right) = \lim_{x \to 1^+} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}}$$

$$= \frac{1}{1 + 1}$$

$$= \frac{1}{2}.$$