THE 1-2-3 CONJECTURE FOR UNIFORM HYPERGRAPHS

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November 14, 2015

Abstract. Given an \(r\)-uniform hypergraph \(H = (V, E)\) and a weight function \(\omega : E \to \{1, \ldots, w\}\), a coloring of vertices of \(H\), induced by \(\omega\), is defined by \(c(v) = \sum_{e \ni v} \omega(e)\) for all \(v \in V\). In this paper, we show that for almost all \(3\)-uniform hypergraphs there exists a weighting of the edges from \(\{1, 2\}\) that induces a proper vertex-coloring (that means with no monochromatic edges). For \(r \geq 4\), we show that almost all \(r\)-uniform hypergraphs can be weighted just from \(\{1\}\). These results extend a previous work of Addario-Berry, Dalal and Reed for graphs. Finally, we also give a sufficient condition for uniform hypergraphs to have a proper coloring induced by a weighting from \(\{1, 2\}\).

1. Introduction

We start with a few definitions. Let \(H = (V, E)\) be a hypergraph on the vertex set \(V\) and with the set of edges \(E \subseteq 2^V\). Let \(\omega : E \to \{1, \ldots, w\}\) be a weight function. Furthermore, let \(c : V \to \mathbb{N}\) be a vertex-coloring induced by \(\omega\) defined as

\[c(v) = \sum_{e \ni v} \omega(e)\]

for each \(v \in V\). The vertex-coloring is proper if no edge is monochromatic. We say that \(H\) is \(w\)-weighted if there exists \(\omega : E \to \{1, \ldots, w\}\) such that the vertex-coloring \(c\) induced by \(\omega\) is proper.

In this paper we mainly deal with \(r\)-uniform hypergraphs. A hypergraph \(H = (V, E)\) is \(r\)-uniform if for each \(e \in E\), \(|e| = r\). (Clearly, a 2-uniform hypergraph is just a graph.)

In 2004 Karoński, Łuczak and Thomason \cite{10} proposed the following conjecture.

1-2-3 Conjecture \cite{10}). Every graph without isolated edges is 3-weighted.

This conjecture attracted a lot of attention and has been studied by several researchers (see, e.g., a survey paper of Seamone \cite{11}). The 1-2-3 conjecture is still open but it is known due to a result of Kalkowski, Karoński, and Pfender \cite{8} that every graph without isolated edges is 5-weighted.

Quite recently, the same authors \cite{9} started to study analogous problems for hypergraphs. In particular, they proved that any \(r\)-uniform hypergraph is \((5r - 5)\)-weighted.
and that 3-uniform hypergraphs are even 5-weighted. The authors also asked whether there is an absolute constant \( w_0 \) such that every \( r \)-uniform hypergraph is \( w_0 \)-weighted. Furthermore, they conjectured that each 3-uniform hypergraph without isolated edges is 3-weighted. In this paper we show that for almost all uniform hypergraphs these conjectures hold.

We say that \textit{almost all} \( r \)-uniform hypergraphs on \( n \) vertices have property \( \mathcal{P} \) if as \( n \) tends to infinity, \( o(2^{r^2}) \) \( r \)-uniform hypergraphs on \( n \) vertices do not have property \( \mathcal{P} \).

Theorem 1.1.

(i) \textit{Almost all} 3-uniform hypergraphs are 2-weighted (but not 1-weighted).

(ii) For a fixed \( r \geq 4 \) \textit{almost all} \( r \)-uniform hypergraphs are 1-weighted.

The second part is easier, since a hypergraph is 1-weighted if and only if there is no edge consisting of vertices that all have the same degree. Thus, the interesting part is showing that almost all 3-uniform hypergraphs are 2-weighted. This theorem (proven in Sections 3 and 4) extends a result of Addario-Berry, Dalal and Reed [1] who showed that almost all graphs are 2-weighted. However, our proofs are completely different.

It is not difficult to see that in general \( r \)-uniform hypergraphs are not 2-weighted. Indeed, define an \( r \)-uniform hypergraph \( H = (V,E) \) on \( V = \{x_1, \ldots, x_r, y_1, \ldots, y_r, z_1, \ldots, z_r\} \) with edges \( e_1 = \{x_1, \ldots, x_r\} \), \( e_2 = \{y_1, \ldots, y_r\} \), \( e_3 = \{z_1, \ldots, z_r\} \), \( f_1 = \{x_2, \ldots, x_r, y_1\} \), \( f_2 = \{y_2, \ldots, y_r, z_1\} \), and \( f_3 = \{z_2, \ldots, z_r, x_1\} \) (see Figure 1). If there is a proper coloring of \( H \) induced by some \( w : E \rightarrow \{1, 2\} \), then for some \( i, j \) we must have \( \omega(e_i) = \omega(e_j) \) and consequently edge \( f_\ell \subseteq V(e_i) \cup V(e_j) \) is monochromatic, a contradiction.

We believe that all uniform hypergraphs without isolated edges are 3-weighted. As a matter of fact we will show that a large class of hypergraphs is 2-weighted. An example of such class consists of all \( r \)-uniform hypergraphs for \( r \geq 5 \) and minimum degree \( \Omega(n^4) \) and pair degree “not too big” (see Section 5).

2. Preliminaries

In order to prove Theorem 1.1 we will consider random hypergraphs. Let \( \mathbb{H}^{(k)}(n,p) \) be an \( r \)-uniform \textit{random hypergraph} such that each of \( \binom{n}{r} \) \( r \)-tuples is contained with
probability $p$, independently of others. We say that an event $E_n$ occurs with high probability, or w.h.p. for brevity, if $\lim_{n \to \infty} \Pr(E_n) = 1$. Hence, to show that almost all $r$-uniform hypergraphs on $n$ vertices have property $\mathcal{P}$ as $n$ tends to infinity it suffices to show that w.h.p. $\mathbb{H}^{(k)}(n, 1/2) \in \mathcal{P}$.

This paper uses some standard probabilistic tools, which we state here for convenience (for more details see, e.g., [2, 7]).

**First Moment Method.** Let $X$ be a nonnegative integral random variable. If $E(X) = o(1)$, then w.h.p. $X = 0$.

**Second Moment Method.** Let $X$ be a nonnegative integral random variable. If $\text{Var}(X) = o(E(X)^2)$, then w.h.p. $X \geq 1$.

Let $\text{Bin}(n, p)$ denotes the random variable with binomial distribution with number of trials $n$ and probability of success $p$.

**Chernoff’s Bound.** If $X \sim \text{Bin}(n, p)$ and $0 < \gamma \leq E(X)$, then
\[
\Pr(\{|X - E(X)| \geq \gamma\}) \leq 2 \exp\left(-\frac{\gamma^2}{3E(X)}\right).
\]

**Bernstein’s Bound.** Let $X_1, \ldots, X_m$ be independent random variables, and $X = \sum_{i=1}^{m} X_i$. Suppose that $|X_i - E(X_i)| \leq C$ always holds for all $i$. Then, for all positive $\gamma$
\[
\Pr(\{|X - E(X)| \geq \gamma\}) \leq 2 \exp\left(-\frac{1}{2} \gamma^2 \frac{\sum_{i=1}^{m} \text{Var}(X_j) + \frac{1}{3} C \gamma}{\sum_{i=1}^{m} \text{Var}(X_j)}\right).
\]

We will also use the union bound.

**Union Bound.** If $E_i$ are events, then
\[
\Pr\left(\bigcup_{i=1}^{m} E_i\right) \leq m \cdot \max\{\Pr(E_i) : i \in [m]\}.
\]

Several times we will also need to estimate binomial coefficients (for more details see, e.g., Chapter 22 in [6]).

**Binomial Coefficients Approximation.**

(A1) Let $k > 0$ and $\ell$ be functions of $m$ ($\ell$ can be negative). Assume that $\ell^2 = o(k)$ and $\ell^2 = o(m - k)$ as $m$ tends to infinity. Then,
\[
\binom{m}{k + \ell} \sim \binom{m}{k} \left(\frac{m - k}{k}\right)^\ell.
\]

(A2) Let $k \geq 1$ be a fixed integer. Then,
\[
\sum_{i=0}^{m} \binom{m}{i}^k \sim \left(2^m \sqrt{\frac{2}{\pi m}}\right)^k \sqrt{\frac{\pi m}{2k}}.
\]

(A3)
\[
\binom{m}{\lfloor m/2 \rfloor} \sim \frac{2^{m+1/2}}{\sqrt{\pi m}}.
\]

All logarithms in this paper are natural (base $e$).
3. Almost all 3-uniform hypergraphs are 2-weighted

First we show by using the second moment method that almost all 3-uniform hypergraphs are not 1-weighted. Recall that this is equivalent to showing the following:

**Proposition 3.1.** Let $\mathbb{H} = \mathbb{H}^{(3)}(n, 1/2)$. Then, w.h.p. there is an edge $\{v_1, v_2, v_3\}$ in $\mathbb{H}$ such that $\text{deg}(v_1) = \text{deg}(v_2) = \text{deg}(v_3)$.

**Proof.** Let $X$ be a random variable that counts the number of edges $\{v_1, v_2, v_3\}$ in $\mathbb{H}^{(3)}(n, 1/2) = ([n], E)$ such that $\text{deg}(v_1) = \text{deg}(v_2) = \text{deg}(v_3)$.

First we compute the expected value of $X$. Fix $e = \{v_1, v_2, v_3\} \in \binom{[n]}{3}$ and condition on $e \in E$. Let $x_i$ be the random variable that counts the number of edges containing only $v_i$ (and neither of the two remaining vertices). Furthermore, let $y_{i,j}$ be the random variable that counts the number of edges containing $v_i$ and $v_j$ (and not the remaining third vertex), where $\{i, j\} \in \binom{[3]}{2}$. Clearly, $x_i \sim \text{Bin} \left(\binom{n-3}{2}, 1/2\right)$ and $y_{i,j} \sim \text{Bin} (n - 3, 1/2)$. The Chernoff bound together with the union bound yield that w.h.p. for any $e, i, j$,

$$\left| x_i - \frac{n - 3}{2} \right| = O(n \sqrt{\log n}) \quad \text{and} \quad \left| y_{i,j} - \frac{n - 3}{2} \right| = O(\sqrt{n \log n}). \quad (1)$$

Clearly, for any $e = \{v_1, v_2, v_3\}$ we have

$$\text{deg}(v_1) = x_1 + y_{1,2} + y_{1,3} + 1,$$
$$\text{deg}(v_2) = x_2 + y_{1,2} + y_{2,3} + 1,$$
$$\text{deg}(v_3) = x_3 + y_{1,3} + y_{2,3} + 1,$$

and so w.h.p. for any $e$ and $i$,

$$\left| \text{deg}(v_i) - \frac{n - 3}{2} \right| = O(n \sqrt{\log n}).$$

Conditioning on $y_{1,2} = \beta_{1,2}$ and $y_{1,3} = \beta_{1,3}$, we get that the probability that $\text{deg}(v_1) = a$ is

$$\left(\binom{n-3}{2} \div (a - \beta_{1,2} - \beta_{1,3} - 1)\right) 2^{-\binom{n-3}{2}}.$$  

Due to $[1]$, we may assume that $\beta_{i,j} = \frac{n}{2} + \ell_{i,j}$ and $|a - \binom{n-3}{2}|/2 = O(n \sqrt{\log n})$, where $|\ell_{i,j}| = O(\sqrt{n \log n})$.

By [A1] (applied with $m = \binom{n-3}{2}$, $k = a - n$, and $\ell = \ell_{1,2} + \ell_{1,3} + 1$) we obtain that

$$\left(\binom{n-3}{2} \div (a - \beta_{1,2} - \beta_{1,3} - 1)\right) = \left(\binom{n-3}{2} \div (a - n - (\ell_{1,2} + \ell_{1,3} + 1))\right)$$
$$\sim \left(\binom{n-3}{2} \div (a - n)\right) \frac{\ell_{1,2} + \ell_{1,3} + 1}{a - n}.$$ 

Now

$$\binom{n-3}{2} \div (a - n) \leq \binom{n-3}{2} \div 2 + O(n \sqrt{\log n})$$
$$\binom{n-3}{2} \div 2 - O(n \sqrt{\log n}) = 1 + O\left(\frac{\log n}{n}\right).$$
and
\[
\frac{(n^3 - 1)(a - n)}{a - n} \geq \frac{(n^3 - 1)/2 - O(n\sqrt{\log n})}{(n^3 - 1)/2 + O(n\sqrt{\log n})} = 1 - O\left(\frac{\log n}{n}\right).
\]
Hence, since \(|\ell_{1,2} + \ell_{1,3} + 1| = O(\sqrt{n\log n})\), we get that
\[
\frac{(n^3 - 1)(a - n)}{a - n} \sim 1,
\]
and consequently,
\[
\left(\frac{(n^3 - 1)(a - n)}{a - n}\right)^{\ell_{1,2} + \ell_{1,3} + 1} \sim 1,
\]
and consequently,
\[
\left(\frac{(n^3 - 1)(a - n)}{a - n}\right)^{\ell_{1,2} + \ell_{1,3} + 1} \sim 1.
\]
Similarly,
\[
\left(\frac{(n^3 - 1)(a - n)}{a - n}\right)^{\ell_{1,2} + \ell_{1,3} + 1} \sim 1.
\]
Thus, conditioning on \(\beta_{i,j}\)'s and \(e \in E\) the probability that \(\deg(v_1) = \deg(v_2) = \deg(v_3) = a\) equals
\[
\left(\frac{(n^3 - 1)(a - n)}{a - n}\right)^{\ell_{1,2} + \ell_{1,3} + 1} \sim 1.
\]
So,
\[
\Pr(\deg v_1 = \deg(v_2) = \deg(v_3), \{v_1, v_2, v_3\} \in E) \sim \frac{1}{2} \sum_{\beta_{i,j}} \Pr(y_{i,j} = \beta_{i,j}) \sum_{a} \left(\frac{(n^3 - 1)(a - n)}{a - n}\right)^{3} 2^{-3(n^3 - 1)}
\]
where the sums are taken over all values of \(\beta_{i,j} = \frac{n}{2} + \ell_{i,j}\) with \(|\ell_{i,j}| = O(\sqrt{n\log n})\) and \(|a - (n^3 - 1)/2| = O(n\sqrt{\log n})\). Since
\[
\sum_{\beta_{i,j}} \Pr(y_{i,j} = \beta_{i,j}) \sim \sum_{\beta_{i,j} = 0} \Pr(y_{i,j} = \beta_{i,j}) = 1
\]
and
\[
\sum_{a} \left(\frac{(n^3 - 1)(a - n)}{a - n}\right)^{3} \sim \sum_{a=0} \left(\frac{(n^3 - 1)(a - n)}{a - n}\right)^{3}
\]
we get
\[
\Pr(\deg v_1 = \deg(v_2) = \deg(v_3), \{v_1, v_2, v_3\} \in E) \sim \frac{1}{2} \sum_{a=0} \left(\frac{(n^3 - 1)(a - n)}{a - n}\right)^{3} 2^{-3(n^3 - 1)} \sim \frac{2}{\pi \sqrt{3n^2}}.
\]
where the latter follows from [A2] (applied with \( m = \binom{n-3}{2} \) and \( k = 3 \)). Thus,

\[
E(X) \sim \left( \frac{n}{3} \right) \frac{2}{\pi \sqrt{3}n^2},
\]

which goes to infinity together with \( n \).

Now we compute the variance. We show that \( E(X^2) \sim (E(X))^2 \). For \( e \in \binom{[n]}{3} \), let \( X_e \) be an indicator random variable which is equal to 1 if all three vertices in \( e \) have the same degree. Thus,

\[
E(X_e) \sim \binom{n-3}{2} \pi \sqrt{\frac{3}{n^2}},
\]

which goes to infinity together with \( n \).

If \( |e \cap f| = 2 \), then we may assume that \( e \cup f = \{v_1, \ldots, v_4\} \) and so similarly to the previous computations

\[
\Pr(X_e X_f = 1) = \Pr(\deg(v_1) = \cdots = \deg(v_4) \text{ and } e, f \in E) \sim \frac{1}{4} \sum_a \left( \binom{n-4}{2} \right)^4 2^{-4(n-4)^4} = O \left( \frac{1}{n^3} \right)
\]

and consequently,

\[
\sum_{|e \cap f| = 2} E(X_e X_f) = O \left( n^4 \cdot \frac{1}{n^3} \right) = O(n) = o((E(X))^2).
\]

Similarly,

\[
\sum_{|e \cap f| = 1} E(X_e X_f) = O \left( n^5 \cdot \frac{1}{n^4} \right) = O(n) = o((E(X))^2).
\]

Finally, we describe how to compute \( \sum_{|e \cap f| = 0} E(X_e X_f) \). Fix two vertex disjoint triples \( e = \{v_1, v_2, v_3\} \) and \( f = \{v_4, v_5, v_6\} \) in \( \binom{[n]}{3} \) and condition on \( e, f \in E \). Similarly as before we define \( x_i \) and \( y_{i,j} \). Furthermore, we define \( z_{i,j,k} \) that counts the number of edges containing 1 vertex from \( e \) and 2 from \( f \) or 2 vertices from \( e \) and 1 from \( f \).

Clearly, \( x_i \sim \text{Bin} \left( \binom{n-6}{2}, 1/2 \right) \) and \( y_{i,j} \sim \text{Bin} \left( n-6, 1/2 \right) \), and \( z_{i,j,k} \) are indicator random variables. Now for example, conditioning on \( e \in E \) we get

\[
\deg(v_1) = x_1 + \sum_{2 \leq j \leq 6} y_{1,j} + \sum_{4 \leq j < k \leq 6} z_{1,j,k} + \sum_{1 \leq j < k \leq 3} z_{1,j,k} + 1.
\]

Hence, in a very similar way one can show that

\[
\sum_{|e \cap f| = 0} E(X_e X_f) \sim \frac{1}{2} \left( \binom{n}{3} \right) \sum_a \left( \binom{n-6}{2} \right)^3 2^{-3(n-6)^6} \cdot \frac{1}{2} \left( \binom{n-3}{3} \right) \sum_b \left( \binom{n-6}{2} \right)^3 2^{-3(n-6)^6} \sim (E(X))^2.
\]

Thus, we are done by the second moment method. \( \square \)
To finish the proof of part (i) of Theorem \ref{thm:1-2-3} we need to show that almost all 3-uniform hypergraphs are 2-weighted. First we need some auxiliary results. For \( G = (V, E) \) and \( S \subseteq V \), let \( N(S) \) denote the \textit{neighborhood of} \( S \), i.e., the set of all vertices in \( V \) adjacent to some element of \( S \).

**Lemma 3.2.** There exists a positive constant \( \gamma \) such that with probability \( 1 - o(1/n) \) the random bipartite graph \( G(n, n, 1/2) \) contains at least \( \gamma n \) edge disjoint perfect matchings.

This lemma is a weaker version of a more general result of Frieze and Krivelevich \cite{frieze} (where the authors obtained an optimal constant \( \gamma = 1/2 - o(1) \)). For the sake of completeness we show here a simple proof.

**Proof.** Let \( G = G(n, n, 1/2) \) be a random bipartite graph on the set of vertices \( A \cup B \), where \( |A| = |B| = n \). Set \( \gamma = 1/10 \).

First observe that since for any \( u, v \in A \) and \( u, v \in B \) we have \( |N(v)| \sim \text{Bin}(n, 1/2) \) and \( |N(\{u, v\})| \sim \text{Bin}(n, 3/4) \), the Chernoff bound yields that with probability \( 1 - o(1/n) \) for any \( u, v \in A \) and \( u, v \in B \),

\[
|N(v)| \geq n/2 - O(\sqrt{n \log n}) \quad \text{and} \quad |N(\{u, v\})| \geq 3n/4 - O(\sqrt{n \log n}).
\]

Assume that we already found a collection \( M_i = \{M_1, \ldots, M_i\} \) of perfect matchings in \( G \). We show that \( G_{i+1} = G \setminus M_i \) also contains a perfect matching \( M_{i+1} \). It suffices to show that if \( i < \gamma n \), then the Hall condition holds, i.e.,

\[
\text{if } S \subseteq A \text{ and } |S| \leq n/2, \text{ then } |N_{M_i}(S)| \geq |S|, \tag{2}
\]

and

\[
\text{if } T \subseteq B \text{ and } |T| \leq n/2, \text{ then } |N_{M_i}(T)| \geq |T|. \tag{3}
\]

Indeed, if \( S = \{v\} \), then

\[
|N_{M_i}(S)| = |N_{M_i}(v)| = |N_G(v)| - i \geq n/2 - O(\sqrt{n \log n}) - \gamma n \geq 1 = |S|.
\]

Therefore, we may assume that \( 2 \leq |S| \leq n/2 \). Let \( \{u, v\} \subseteq S \). Then,

\[
|N_{M_i}(S)| \geq |N_{M_i}(\{u, v\})| \geq |N_G(\{u, v\})| - 2i \geq 3n/4 - O(\sqrt{n \log n}) - 2\gamma n \geq n/2 \geq |S|
\]

and \( (2) \) holds. Similarly, \( (3) \) holds, too. \( \square \)

**Lemma 3.3.** Let \( H \) be a 3-partite 3-uniform random hypergraph on vertex set \( V_1 \cup V_2 \cup V_3 \) with \( |V_1| = |V_2| = |V_3| = n \) and probability \( 1/2 \). Then, there exists a positive constant \( \gamma \) such that w.h.p. \( H \) contains at least \( \gamma n^2 \) edge disjoint perfect matchings.

**Proof.** Consider first a complete bipartite graph \( F \) on \( V_1 \cup V_2 \). Then, since \( F \) is \( n \)-regular bipartite graph, it can be decomposed into \( n \) edge disjoint perfect matchings, say \( F = M_1 \cup \cdots \cup M_n \). Let \( H_i \) be a 3-partite 3-uniform hypergraph on \( V_1 \cup V_2 \cup V_3 \) and the set of edges

\[
E_i = \{e \cup v : e \in M_i \text{ and } v \in V_3\}.
\]

Thus, the complete 3-partite 3-uniform hypergraph on \( V_1 \cup V_2 \cup V_3 \) is the edge disjoint union of \( H_i \)'s over all \( 1 \leq i \leq n \). We generate a random 3-partite 3-uniform hypergraph by revealing edges in each \( H_i \). It suffices to show that the random hypergraph induced by \( H_i \) contains with probability \( 1 - o(1/n) \) at least \( \gamma n \) edge disjoint perfect matchings, where \( \gamma \) is a constant from Lemma 3.2. The latter is obviously true since the random
graph induced by $H_i$ can be viewed as $G = G(n, n, 1/2)$ on $M \cup V_3$ (since $|M| = |V_3| = n$) and any perfect matching in $G$ yields a perfect matching in the 3-partite hypergraph.

Now we are ready to finish the proof of Theorem 1.1(i).

**Proposition 3.4.** $\mathbb{H}^{(3)}(n, 1/2)$ is 2-weighted w.h.p.

**Proof.** Our strategy is as follows. First we consider an equipartition $V_1 \cup V_2 \cup V_3$ of the vertex set $V = [n]$. Then we reveal the random hypergraph $\mathbb{H}^{(3)}(n, 1/2)$. Due to Lemma 3.3 (assuming that $n$ is divisible by 3) there is w.h.p. a family $\mathcal{M}$ of disjoint perfect matchings such that $\mathcal{M} = [\gamma n^2]$ and each edge in each matching has one vertex in each part of the partition. We then randomly label the vertices in $V_1$ with real number labels in $[0, 1/9]$, the vertices in $V_2$ with labels in $[1/9, 2/9]$, and the vertices in $V_3$ with labels in $[2/9, 1/3]$. By $\ell(v)$ we denote the label of vertex $v$. We then randomly weight the edges, where any non-matching edge $\{u, v, w\} \in \binom{\gamma n}{3}$ gets weight 2 with probability $\ell(u) + \ell(v) + \ell(w)$, and any matching edge gets weight 2 with probability $\frac{1}{2}$. We will show that

$$E(|\{c: 0 \leq c \leq n^2, \exists x, y, z \text{ such that } c(x) = c(y) = c(z) = c\}|)$$

$$\leq \sum_{c=0}^{n^2} E(|\{\{x, y, z\}: c(x) = c(y) = c(z) = c\}|) = o(1),$$

which will complete the proof, since by the first moment method w.h.p. no three vertices get the same color.

In this paragraph and the next we are revealing only the random hypergraph $H = \mathbb{H}^{(3)}(n, 1/2)$, and all probabilities are with respect to this distribution. The degrees of vertices are concentrated by the Chernoff bound. More specifically, for any vertex $v$, and any distinct parts $V_i$ and $V_j$ there are $m = n^2/9 + O(n)$ triples containing $v$ together with one additional vertex from $V_i$ and one from $V_j$ (the $O(n)$ term is to account for the possibility that $v$ is in one of $V_i$ or $V_j$). Each of these triples has probability $1/2$ of being an edge of $H$, independently, and so if we let $d_{i,j}(v)$ denote the number of these edges present in $H$, the Chernoff bound tells us that

$$\Pr(|d_{i,j}(v) - m/2| > n \log n) \leq 2 \exp \left( -\frac{(n \log n)^2}{3m/2} \right) = \exp \left( -\Omega(\log^2 n) \right)$$

and so the union bound gives us that the probability there exists $v, i, j$ such that $|d_{i,j}(v) - m/2| > n \log n$ is at most

$$3n \cdot \exp \left\{ -\Omega(\log^2 n) \right\} = o(1).$$

Hence, w.h.p. for every vertex $v$ and distinct parts $V_i, V_j$ there are $n^2/18 + O(n \log n)$ edges containing $v$ and additionally one vertex from each of $V_i, V_j$. Similarly (using the Chernoff and union bounds), w.h.p. for any vertex $v$ and part $V_i$ there are $n^2/36 + O(n \log n)$ edges containing $v$ and additionally two vertices from $V_i$.

For a while we will assume that $n$ is divisible by 3. Now due to Lemma 3.3 there is w.h.p. a family $\mathcal{M}$ of disjoint perfect matchings where $|\mathcal{M}| = \lfloor \gamma n^2 \rfloor$ and each edge in each matching goes between the partition (i.e. has one vertex in each $V_i$). Henceforth
we assume that all the edges of \( H \) were revealed obtaining a hypergraph \( H \) that has the degree properties mentioned in the previous paragraphs, and the family of matchings \( \mathcal{M} \).

Next we reveal the vertex labels and the non-matching edge weights. The weight of edge \( \{x, y, z\} \) is distributed as

\[
w(x, y, z) = \begin{cases} 
1 & \text{with probability } 1 - \ell(x) - \ell(y) - \ell(z) \\
2 & \text{with probability } \ell(x) + \ell(y) + \ell(z)
\end{cases}
\]

so

\[
E(w(x, y, z) | \ell(x), \ell(y), \ell(z)) = 1 + \ell(x) + \ell(y) + \ell(z)
\]

and therefore if by \( c_{H \setminus \mathcal{M}}(x) \) we denote the sum of the weights of non-matching edges containing \( x \), and say we are given only the label \( \ell(x) \), and the hypergraph \( H \) with family of matchings \( \mathcal{M} \), and \( x \in V_i \) then we have

\[
E(c_{H \setminus \mathcal{M}}(x) \mid \ell(x), H) = \sum_{\{x,v,w\} \in E(H) \setminus \mathcal{M}} E(w(x, v, w) \mid \ell(x))
\]

\[
= \sum_{\{x,v,w\} \in E(H) \setminus \mathcal{M}} 1 + \ell(x) + E(\ell(v)) + E(\ell(w))
\]

\[
= (1 + \ell(x)) \deg_{H \setminus \mathcal{M}}(x) + \sum_{\{x,v,w\} \in E(H) \setminus \mathcal{M}} E(\ell(v)) + E(\ell(w))
\]

\[
= \ell(x) \left( \frac{1}{4} - \gamma \right) n^2 + \Theta(n^2),
\]

where on the last line we have used our estimate of \( \deg_{H \setminus \mathcal{M}}(x) \). Note that the \( \Theta(n^2) \) term may depend on \( i \) (recall \( x \in V_i \)) but does not otherwise depend on \( x \) or \( \ell(x) \) (by the fact that the degree of each vertex \( v \) into sets \( V_i, V_j \) etc. is concentrated).

Now since \( c_{H \setminus \mathcal{M}}(x) = \sum_{H \setminus \mathcal{M}} w(x, v, w) \), and the random variables \( w(x, v, w) \) are independent (given the vertex labels), we can apply Bernstein’s inequality. For our application we use \( m = \deg_{H \setminus \mathcal{M}}(x) \). We can easily use \( C = 2 \), and put \( \Var(w(x, v, w)) \leq E(w(x, v, w)^2) \leq 4 \). We will set \( \gamma = n \log n \). Then we get

\[
\Pr \left( \left| c_{H \setminus \mathcal{M}}(x) - E(c_{H \setminus \mathcal{M}}(x) \mid \ell(x), H) \right| > n \log n \right) \leq 2 \exp \left( -\frac{\frac{1}{2}n^2 \log^2 n}{4 \deg_{H \setminus \mathcal{M}}(x) + \frac{1}{3} \cdot 2n \log n} \right)
\]

\[
= \exp \left\{ -\Omega \left( \log^2 n \right) \right\}
\]

and so by the union bound, \( w.h.p. \) for each vertex \( x \), \( c_{H \setminus \mathcal{M}}(x) \) is within \( n \log n \) of its expectation.

Now for any fixed integer \( a \) and fixed vertex \( x \),

\[
\Pr \left( \left| \ell(x) \left( \frac{1}{4} - \gamma \right) n^2 - a \right| \leq n \log n \right) = O(\log n/n),
\]

since this is the probability that \( \ell(x) \) falls within an interval of length \( O(\log n/n) \). Since the vertex labels are independent, the probability that there are \( \log^2 n \) many vertices \( x \) with \( \left| \ell(x) \left( \frac{1}{4} - \gamma \right) n^2 - a \right| \leq n \log n \) is at most
(n^2 \log^2 n) \cdot (O(\log n/n))^{\log^2 n} \leq \left( \frac{n e}{\log^2 n} \right)^{\log^2 n} (O(\log n/n))^{\log^2 n} \\
= \left( O \left( \frac{1}{\log n} \right) \right)^{\log^2 n} = o(n^2).

Therefore, by the union bound over integers $c$ from 0 to $n^2$, we have that for all such $c$ the number of vertices $x$ with $|c_{H \setminus M}(x) - c| \leq n \log n$ is at most $\log^2 n$. Henceforth we assume that the labels and non-matching edge weights have been revealed and they satisfy this property.

Now we reveal the matching edge weights. These are 1 or 2 with probability 1/2. We will denote by $c_M(x)$ the sum of the weights of matching edges adjacent to $x$; note that $c(x) = c_M(x) + c_{H \setminus M}(x)$. The key fact we will use here is that since $|M| = \lfloor \gamma n^2 \rfloor$, $c_M(x)$ is not likely to be any one particular value. Indeed, since $c_M(x) - |M| \sim \text{Bin}(|M|, 1/2)$, the mode of $c_M(x)$ occurs with probability

$$
\left( \frac{|M|}{\lfloor |M|/2 \rfloor} \right)^{2-|M|} \sim \frac{\sqrt{2/\gamma \pi}}{n} = O(1/n),
$$

where the latter follows from (A3). Also, w.h.p. for all $x$ we have $|c_M(x) - 3/2 \gamma n^2| \leq n \log n$. Therefore,

$$
\sum_{c=0}^{n^2} E(|\{x, y, z : c(x) = c(y) = c(z) = c\}|) \leq o(1) + O(n^2 \cdot (\log^2 n)^3 \cdot (1/n^3)) = o(1)
$$

on the last line we get the $(1/n)^3$ in the big-O term since the probability that $c(x) = c$ is $O(1/n)$, and conditioning on that event, the event that $c(y) = c$ still has probability $O(1/n)$ (since revealing $c(x)$ only reveals the weights of $O(n)$ of the $\lfloor \gamma n^2 \rfloor$ matching edges containing $y$, $c(y)$ still essentially has the same distribution as it did before conditioning) and similarly for the conditional probability that $c(z) = c$.

If $n$ is not divisible by 3, then we can still use an equipartition $V_1, V_2, V_3$, and a family of (not quite perfect) matchings $\mathcal{M}$ where $|M| = \gamma n^2$, each $M \in \mathcal{M}$ has size $[n/3]$, and each vertex is covered by $\gamma n^2 - O(n)$ matchings in $\mathcal{M}$. \qed

4. Almost all $r$-uniform hypergraphs with $r \geq 4$ are 1-weighted

**Proposition 4.1.** Let $r \geq 4$ and $\mathbb{H} = \mathbb{H}^{(r)}(n, 1/2)$. Then, w.h.p. $\mathbb{H}$ contains no four vertices $v_1, v_2, v_3$, and $v_4$ such that $\deg(v_1) = \cdots = \deg(v_4)$.

**Proof.** Let $X$ be the random variable that counts the number of quadruples $\{v_1, v_2, v_3, v_4\}$ in $\mathbb{H}$ such that $\deg(v_1) = \cdots = \deg(v_4)$. Fix $v_1, v_2, v_3$, and $v_4$. Let $x_i$ be the random variable that counts the number of edges containing $v_i$ and no other vertex from $\{v_1, \ldots, v_4\}$, where $i = 1, 2, 3, 4$. Let $y_{i,j}$ be the random variable that counts the number of edges containing $v_i$ and $v_j$ (and no other remaining vertices), where $\{i, j\} \in \binom{[4]}{2}$. Similarly we define $z_{i,j,k}$ as the random variable that counts number of edges containing $v_i, v_j$ and
Furthermore, due to our assumption on $\deg(u,v)$ and for any $u,v$

$$x_1 + y_{1,2} + y_{1,3} + y_{1,4} + z_{1,2,3} + z_{1,2,4} + z_{1,3,4} =$$

$$x_2 + y_{1,2} + y_{2,3} + y_{2,4} + z_{1,2,3} + z_{1,2,4} + z_{2,3,4} =$$

$$x_3 + y_{1,3} + y_{2,3} + y_{3,4} + z_{1,2,3} + z_{1,3,4} + z_{2,3,4} =$$

$$x_4 + y_{1,4} + y_{2,4} + y_{3,4} + z_{1,2,4} + z_{1,3,4} + z_{2,3,4}.$$  

Furthermore, $x_i \sim \text{Bin} \left( \binom{n-4}{r-1}, 1/2 \right)$, $y_{i,j} \sim \text{Bin} \left( \binom{n-4}{r-2}, 1/2 \right)$, and $z_{i,j,k} \sim \text{Bin} \left( \binom{n-4}{r-3}, 1/2 \right)$. Now similarly as in Section 3 one can show that

$$E(X) \sim \left( \frac{n}{4} \right) \sum_a \left( \frac{\binom{n-4}{a}}{r-1} \right) 4^{\binom{a}{r-1}} = O(n^{4-3(r-1)/2}) = o(1),$$

for any $r \geq 4$. Thus, the first moment method yields the statement. \hfill \Box

5. A CLASS OF 2-WEIGHTED $r$-UNIFORM HYPERGRAPHS

In this section we give a sufficient condition for a hypergraph to be 2-weighted. For a hypergraph $H = (V, E)$ and two vertices $u$ and $v$ in $V$ denote by $\deg(u,v)$ the pair degree which is the number of edges in $E$ containing $\{u, v\}$.

**Theorem 5.1.** Let $H = (V, E)$ be an $r$-uniform hypergraph of order $n$ with maximum degree $\Delta$ satisfying

$$\Delta \cdot \sum_{\{x_1, \ldots, x_k\} \in E(H)} \prod_{i=1}^r \frac{1}{\deg(x_i)} = o(1),$$

and for any $u, v \in V$

$$\deg(u, v) = o(\deg(u)),$$

where the asymptotic is taken in $n$. Then, $H$ is 2-weighted.

**Proof.** To each edge we assign 1 or 2 with probability 1/2. Since the mode of $c(x)$ occurs with probability $\sqrt{2/\pi} / \sqrt{\deg(x)}$, we get

$$\Pr(c(x_1) = \cdots = c(x_r) = c) = \prod_{i=1}^r \Pr(c(x_i) = c \mid c(x_1) = \cdots = c(x_{i-1}) = c)$$

$$\leq \prod_{i=1}^r \frac{\sqrt{2/\pi}}{\sqrt{\deg(x_i) - o(\deg(x_i))}}$$

due to our assumption on $\deg(u, v) = o(\deg(u))$. Thus,

$$\mathbb{E}(|\{c : 0 \leq c \leq 2\Delta, \exists \{x_1, \ldots, x_r\} \text{ such that } c(x_1) = \cdots = c(x_r) = c\}|)$$

$$\leq \sum_{c=1}^{2\Delta} \sum_{\{x_1, \ldots, x_k\} \in E(H)} \Pr(c(x_1) = \cdots = c(x_r) = c)$$

$$\leq 2\Delta \sum_{\{x_1, \ldots, x_k\} \in E(H)} \prod_{i=1}^r \frac{\sqrt{2/\pi}}{\sqrt{\deg(x_i) - o(\deg(x_i))}} = o(1),$$
by assumption completing the proof.

**Corollary 5.2.** Let $H = (V,E)$ be an $r$-uniform hypergraph of order $n$ and size $m$ with maximum degree $\Delta$ and minimum degree $\delta$ satisfying $\Delta \cdot m \cdot \left(\frac{1}{\delta}\right)^{k/2} = o(1)$ and $\text{deg}(u,v) = o(\text{deg}(u))$ for any $u,v \in V$. Then, $H$ is 2-weighted.

Since trivially $\Delta = O(n^{r-1})$ and $m = O(n^k)$ we get for $\delta = \Omega(n^4)$ that
\[
\Delta \cdot m \cdot \left(\frac{1}{\delta}\right)^{k/2} = O(n^{k-1} \cdot n^k \cdot 1/n^{2k}) = o(1).
\]

Consequently the following holds.

**Corollary 5.3.** Let $r \geq 5$. Let $H = (V,E)$ be an $r$-uniform hypergraph of order $n$ with minimum degree $\Omega(n^4)$ such that $\text{deg}(u,v) = o(\text{deg}(u))$ for any $u,v \in V$. Then, $H$ is 2-weighted.

### 6. Concluding remarks

In view of results of this paper we conjecture that for any $r \geq 3$, any $r$-uniform hypergraph without isolated edges is 3-weighted. If true, then this is clearly best possible (see Figure 1). We also believe that determining whether a particular uniform hypergraph is 2-weighted is NP-complete. The latter would extend a similar result for graphs that was independently obtained by Dehghan, Sadeghi and Ahadi [3], and Dudek and Wajc [4].

It might be also of some interest to study the weightiness of $H^{(r)}(n,p)$ for any $p = p(n)$. It is not difficult to see that the proof of Theorem 1.1 can be modified by replacing $1/2$ by any constant $p \in (0,1)$ obtaining the same conclusion.

### References


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