LECTURE 6 EXERCISES

1 : Do exercise 4.3.11 on page 140 of the Dembo and Zeitouni book. Show that these results imply

$$\lim_{M\uparrow\infty} \inf_{\{x:\phi(x)>M\}} (I(x)-\phi(x)) = \infty$$

2 : Extend the results of exercise 4.3.11 to functions ϕ which can take the value $-\infty$. This is an important extenxtion because it enables asymptotic analysis of quantities like

$$E_P\left[\Phi(Z_\varepsilon)^{1/\varepsilon}\right] = E_P\left[\exp\left(\frac{1}{\varepsilon}\log\Phi(Z_\varepsilon)\right)\right]$$

where the function Φ is non-negative, but may take the value 0.

2 : (Theorem $III.17$ on page 34 of the Hollander book) Let $\mathcal{X}$ be a Polish space with Borel σ-algebra $\mathcal{B}_\mathcal{X}$. Let the measures $(\mathbb{P}_n)_{n\in\mathbb{N}}$ satisfy a LDP on $\mathcal{X}$ with good rate function I. Let $F:\mathcal{X}\mapsto\mathbb{R}$ be a continuous function bounded from above. For each n define the set function

$$J_n(S) = \int_S e^{nF(x)} P_n(dx) \qquad S\in\mathcal{B}_\mathcal{X}$$

and the probability measures $\left(\mathbb{P}_n^F\right)_{n\in\mathbb{N}}$ via

$$\mathbb{P}_n^F(S) = \frac{J_n(S)}{J_n(X)} \qquad S\in\mathcal{B}_\mathcal{X}$$

Define the rate function

$$I^F(x) = \sup_{y\in\mathcal{X}} (F(y)-I(y)) - (F(x)-I(x))$$

Show that I^F is a good rate function and that $\left(\mathbb{P}_n^F\right)_{n\in\mathbb{N}}$ satisfy a LDP on $\mathcal{X}$ with I^F.

4 : In this exercise you will show that the Laplace Principle implies a LDP for measures on Polish spaces. Thus, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let $(Z_n)_{n\in\mathbb{N}}$ be a family of Borel measurable random variables taking values in a Polish space $\mathcal{X}$ (with Borel σ-algebra $\mathcal{B}_\mathcal{X}$) with distributions $(\mu_n)_{n\in\mathbb{N}}$.

Prove that if there exists a good rate function $I:\mathcal{X}\mapsto[0,\infty]$ such that for all continuous and bounded functions $\phi:\mathcal{X}\mapsto\mathbb{R}$ the following limit holds (the minus sign is for ease of notation only)

$$\lim_{n\uparrow\infty}\frac{1}{n}\log E_P\left[\exp\left(-n\phi(Z_n)\right)\right] = \inf_{x\in\mathcal{X}}(\phi(x)+I(x))$$

Then $(\mu_n)_{n\in\mathbb{N}}$ solves the LDP with good rate function I.

Hints : note that $\mu_n(A) = E_P\left[\exp\left(-n\phi(X_n)\right)\right]$ for the "indicator" function

$$\phi(x) = \begin{cases} 0 & x\in A \\ \infty & x\notin A \end{cases}$$

This function is not continuous, however you can approximate it with continuous bounded functions. For the upper bound, if F is closed then set $\phi_j(x) = j\left(d(x,F)\wedge 1\right)$. Recall that the lower bound follows if for all $x \in \mathcal{X}$ with $I(x) < \infty$ there is a δ small enough so that

$$\liminf_{n\uparrow\infty} \frac{1}{n} \log \mu_n \left(B(x,\delta)\right) \geq -I(x)$$

To this end, let $\delta > 0$ and set $\phi_j(y) = j\left(\frac{d(x,y)}{\delta} \wedge 1\right)$.