Static Fund Separation of Long Term Investments

Paolo Guasoni
Boston University and Dublin City University
guasoni@bu.edu

Scott Robertson
Carnegie Mellon University
scottrob@andrew.cmu.edu

March 7, 2011

Abstract

This paper proves a class of *static fund separation* theorems, valid for investors with a long horizon and constant relative risk aversion, and with stochastic investment opportunities.

An optimal portfolio decomposes as a constant mix of a few preference-free funds, which are common to all investors. The weight in each fund is a constant that may depend on an investor's risk aversion, but not on the state variable, which changes over time. Vice versa, the composition of each fund may depend on the state, but not on the risk aversion, since a fund appears in the portfolios of several investors.

These results are proved for two classes of models with a single state variable, and several assets with constant correlations with the state. In the *linear* class, the state is an Ornstein-Uhlenbeck process, risk premia are affine in the state, while volatilities and the interest rate are constant. In the *square root* class, the state follows a square root diffusion, expected returns are affine in the state, while volatilities and interest rates are respectively linear and affine in the square root of the state.

Mathematics Subject Classification: (2010) 91G10, 91G80.

JEL Classification: G11, G23.

Keywords: Portfolio Choice, Fund Separation, Long Horizon.

Partially supported by the National Science Foundation under grants DMS-0532390 and DMS-0807994. We thank Bernard Dumas and Hao Xing for useful comments.

1 Introduction

Fund separation means that investors need to trade only a few well-chosen funds, not all the securities in the market. Like primary colors, which mix to span the visible spectrum, these funds, combined in varying weights, span the optimal portfolios of all investors. The idea is as simple as it is important, and its implications range from equilibrium asset pricing to the theory of financial intermediation.

There are two main streams of results[1]. Two-fund separation holds if the risk-free asset and the market span all optimal portfolios, and in equilibrium leads to the Capital Asset Pricing Model. The main assumption of two-fund separation is that investment opportunities are either deterministic or unhedgeable. Then, investors with constant relative risk aversion (CRRA) divide their wealth across funds in proportions that are constant over time, or *static*.

A more complex type of fund separation obtains if investment opportunities are stochastic and hedgeable. If they depend on k state variables, as in Merton (1973), $k+2$ funds are needed to span optimal portfolios. The k funds, in addition to the safe asset and the myopic portfolio, mimic each state variable over time. Then, even CRRA investors need to dynamically change their proportions in the $k+2$ funds, according to complex and generally unknown trading strategies. This is *dynamic fund separation*.

This paper proves *static fund separation* theorems, valid for investors with a long horizon and constant relative risk aversion, and for investment opportunities depending on a single state variable. Unlike two-fund separation theorems, we allow for partially hedgeable investment opportunities, and obtain three or more spanning funds. Unlike the classical $k+2$ separation, we obtain a static decomposition, whereby each investor holds the funds in constant proportions.

Our setting of a single state variable ($k = 1$) already implies dynamic fund separation with three funds: the safe asset, the myopic portfolio, and the hedging portfolio. But only the myopic proportion is constant in this decomposition, while the risk-free and hedging proportions depend on the residual horizon and on the state variable. This dependence severely limits the significance of dynamic separation, and calls for a more precise description of optimal portfolios.

With static fund separation, optimal portfolios are weighted sums of a fixed number of funds, which are common to all investors. The weight of each fund is constant: it may depend on the risk aversion and market parameters, but not on the state variable, which changes over time. Vice versa, the composition of each fund can depend on the state variable and on the market parameters, but not on the risk aversion, since the same fund appears in the portfolios of different investors. Because static fund separation requires constant fund weights, in general it implies more funds than dynamic fund separation. In addition, the number of static funds depends on the model considered, and so do fund weights.

We prove two static separation results, corresponding to two mainstream classes of models. Their common features are a single state variable and several assets, with constant correlations with the state. In the *linear* class, the state is an Ornstein-Uhlenbeck process, risk premia are affine in the state, while volatilities and the interest rate are constant. In the *square root* class, the state follows Feller's (1951) square root diffusion, expected returns are affine in the state, while volatilities and interest rates are respectively linear and affine in the square root of the state. These classes nest several models in the literature, including the stochastic volatility model of Heston (1993), the interest rate model of Cox, Ingersoll and Ross (1985), and the models with

[1]Tobin (1958) first derives two-fund separation in a mean variance setting, while Cass and Stiglitz (1970) and Ross (1978) derive it under assumptions on preferences and return distributions respectively. Chamberlain (1988) and Khanna and Kulldorff (1999) find separating conditions in diffusion models, while Schachermayer, Sîrbu and Taflin (2009) characterize it in a semimartingale setting.

time-varying returns of Kim and Omberg (1996) and Wachter (2002) among others.

While our static fund separation results are based on the joint assumption of CRRA preferences and long horizons, their implications have a broader scope, in both directions. First, Brandt (1999), Barberis (2000) and Wachter (2002) report that optimal portfolios for a ten-year horizon are already close to their long run limit. Second, turnpike theorems[2] suggest that optimal CRRA portfolios are approximately optimal for a broad range of utility functions, at least at long horizons.

The rest of the paper is organized as follows: section 2 describes the general framework, and outlines the main ideas behind our static fund separation results. These ideas are made precise in sections 3 and 4, respectively for the linear and square root models. In each model, static fund separation holds with four funds. We find explicit formulas for both the funds and their weights, and display their composition for typical values of risk aversion, using the market parameters estimated by Barberis (2000). Section 5 concludes, and all proofs are in the Appendix.

2 Problem and Heuristic Solution

This section describes the general setting of the paper, and shows the steps which unify the main results, leaving aside the technical details, which differ across models. Here arguments are presented at a heuristic level, while the next sections contain their precise versions for two classes of models.

2.1 Market and Preferences

The market has a safe asset and several risky assets. Investment opportunities are modeled by a single state variable Y, which drives the safe rate r, the excess returns μ, and the volatility matrix σ. In summary, the prices of the safe asset S^0 and risky assets $S^1, \dots, S^n$ follow the processes:

$$\frac{dS_t^0}{S_t^0} = r(Y_t)dt \tag{2.1}$$

$$\frac{dS_t^i}{S_t^i} = r(Y_t)dt + dR_t^i \qquad 1 \le i \le n \tag{2.2}$$

where the cumulative excess returns $R = (R^1, \dots, R^n)$ and the state variable Y follow the diffusion:

$$dR_t^i = \mu_i(Y_t)dt + \sum_{j=1}^n \sigma_{ij}(Y_t)dZ_t^j \qquad 1 \le i \le n \tag{2.3}$$

$$dY_t = b(Y_t)dt + a(Y_t)dW_t \tag{2.4}$$

$$\langle Z^i, W\rangle_t = \rho_i dt \qquad 1 \le i \le n \tag{2.5}$$

$Z = (Z^1, \dots, Z^n)$ and W are Brownian Motions, and $\rho = (\rho_1, \dots, \rho_n)$ denotes their vector of cross correlations. $\Sigma = \sigma\sigma' = d\langle R, R\rangle_t/dt$ defines the covariance matrix of returns, while $A = a^2 = d\langle Y, Y\rangle_t/dt$ is the variance rate of the state variable, and $\Upsilon = \sigma\rho a = d\langle R, Y\rangle_t/dt$ is the covariation rate between asset returns and the state variable. (The prime sign denotes matrix transposition.) It is assumed there is an open interval $E = (\alpha, \beta)$ with $-\infty \le \alpha < \beta \le \infty$ such that the state variable remains with E at all times. Examples of such intervals are $E = (-\infty, \infty)$ for the linear model of section 3 and $E = (0, \infty)$ for the square root model of section 4. The market defined above is potentially incomplete, because the state variable is not perfectly spanned by asset returns.

[2] For turnpike theorems, see Leland (1972); Hakansson (1974); Huberman and Ross (1983); Cox and Huang (1992); Huang and Zariphopoulou (1999) and Dybvig, Rogers and Back (1999).

An investor trades in the market according to a portfolio $\pi = (\pi_t)_{t\geq 0}$, which represents the proportions of wealth in each risky asset. Since the investor observes the state variable Y and the asset returns R, the portfolio π is adapted to the augmentation of the filtration generated by (R, Y), and is R-integrable. The corresponding wealth process $X^\pi = (X^\pi_t)_{t\geq 0}$ satisfies:

$$\frac{dX^\pi_t}{X^\pi_t} = r(Y_t)dt + \pi'_t dR_t \tag{2.6}$$

Since a positive initial capital $X_0 \geq 0$ implies a positive wealth at all times ($X^\pi_t \geq 0$ a.s. for all $t \geq 0$), doubling strategies are excluded. Investors' preferences display constant relative risk aversion (CRRA), so that their marginal utilities are defined by:

$$U'(x) = x^{p-1} \qquad p < 1 \tag{2.7}$$

For a fixed planning horizon $T > 0$ and a current time $0 \leq t < T$, the investor's goal is to maximize utility from terminal wealth, given the current wealth x and the current state variable y:

$$\max_{(\pi_u)_{u\geq t}} \frac{1}{p} E\left[(X^\pi_T)^p \; |Y_t = y, X_t = x\right] \tag{2.8}$$

2.2 Stochastic Control

Start by defining the value function $V(t, X_t, Y_t)$ as:

$$V(t, X_t, Y_t) = \sup_{(\pi_u)_{u\geq t}} \frac{1}{p} E\left[(X^\pi_T)^p \; |Y_t, X_t\right] \tag{2.9}$$

Following usual control arguments, (2.9) leads to the Hamilton-Jacobi-Bellman equation:

$$V_t + bV_y + \frac{a^2}{2} V_{yy} + rxV_x + \sup_\pi \left(\pi'(\mu V_x + \Upsilon V_{xy})x + \frac{x^2 V_{xx}}{2} \pi' \Sigma \pi \right) = 0 \tag{2.10}$$

with the terminal condition $V(T, x, y) = x^p/p$. Here subscripts denote partial derivatives, and $a, b, \mu, \Sigma, \Upsilon$ are functions of y, although their dependence is omitted to simplify notation. Because V is concave in x, and recalling that $\sup_\pi(\pi' b + \frac{1}{2}\pi' A \pi) = -\frac{1}{2} b' A^{-1} b$ for A negative definite, the equation becomes:

$$V_t + bV_y + \frac{a^2}{2} V_{yy} + rxV_x - (\mu V_x + \Upsilon V_{xy})' \frac{\Sigma^{-1}}{2V_{xx}} (\mu V_x + \Upsilon V_{xy}) = 0 \tag{2.11}$$

and the corresponding optimal portfolio is $\pi = -\Sigma^{-1}\mu \frac{V_x}{xV_{xx}} - \Sigma^{-1}\Upsilon \frac{V_{xy}}{xV_{xx}}$. Since power utility is homothetic, i.e. $U(cx) = c^p U(x)$, and payoffs can be scaled arbitrarily, the value function is also homothetic. Thus, writing $V(t, x, y) = u(t, y) x^p/p$, the HJB equation in terms of the reduced value function u becomes:

$$u_t + (b - q\Upsilon' \Sigma^{-1} \mu) u_y + \frac{a^2}{2} u_{yy} + \left(pr - \frac{q}{2} \mu' \Sigma^{-1} \mu \right) u - q \frac{a^2}{2} \rho' \rho \frac{u_y^2}{u} = 0 \tag{2.12}$$

where $q = p/(p-1)$, and the terminal condition is now $u(T, y) = 1$. The optimal portfolio similarly reduces to:

$$\pi(t, y) = \frac{1}{1-p} \left(\Sigma^{-1}\mu + \Sigma^{-1}\Upsilon \frac{u_y}{u} \right) \tag{2.13}$$

which is the traditional *dynamic* three-fund separation into the safe asset, the myopic portfolio $\Sigma^{-1}\mu$, and the inter-temporal hedging component $\Sigma^{-1}\Upsilon$. The word *dynamic* refers to the dependence of the inter-temporal weight $\frac{u_y}{u}$ on the value function, which entails a complex dynamic trading strategy, depending jointly on the horizon, the state variable, and the risk aversion.

Equation (2.12) simplifies further using the assumption that $\rho'\rho$ is constant. Then, the power substitution of Zariphopoulou (2001) makes this equation linear. Setting $u(t,y) = v(t,y)^\delta$, where $\delta = 1/(1-q\rho'\rho)$, (2.12) turns into a linear parabolic equation for v:

$$\begin{aligned} v_t + \frac{1}{2}Av_{yy} + (b - q\Upsilon'\Sigma^{-1}\mu)v_y + \frac{1}{\delta}\left(pr - \frac{q}{2}\mu'\Sigma^{-1}\mu\right)v = 0 \qquad & (t,y)\in(0,T)\times E \\ v(T,y) = 1 \qquad & y \in E \end{aligned} \tag{2.14}$$

and the optimal portfolio accordingly reduces to:

$$\pi(t,y) = \frac{1}{1-p}\left(\Sigma^{-1}\mu + \delta\Sigma^{-1}\Upsilon\frac{v_y}{v}\right) \tag{2.15}$$

By now, these steps have become standard, and underlie virtually all explicit solutions of portfolio choice problems.

2.3 Eigenvalue Representation

We exploit an eigenvalue expansion, in which the the principal eigenvalue and its eigenvector drive the long horizon limit. The first step in this direction is to rewrite (2.14) in self-adjoint form. Set:

$$b^\nu = b - q\Upsilon'\Sigma^{-1}\mu \qquad V = \frac{1}{\delta}\left(pr - \frac{q}{2}\mu'\Sigma^{-1}\mu\right) \tag{2.16}$$

and define the differential operator:

$$L^\nu = \frac{1}{2}A\frac{d^2}{dy^2} + b^\nu\frac{d}{dy} \tag{2.17}$$

so that (2.14) becomes:

$$\begin{aligned} v_t + L^\nu v + Vv = 0 \qquad & (t,y)\in(0,T)\times E \\ v(T,y) = 1 \qquad & y \in E \end{aligned} \tag{2.18}$$

To ease presentation for functions $f(y)$ of the state variable alone, the symbols $\dot{f}, \ddot{f}$ replace f_y, f_{yy}, the partial derivatives with respect to the state y. Define m as the solution to the ODE

$$\frac{\dot{m}}{m} + \frac{\dot{A}}{A} = \frac{2b^\nu}{A} \tag{2.19}$$

If m integrable, it can be normalized to unit mass, and represents the steady-state distribution of the state variable Y under the equivalent probability measure associated to L^ν. The self-adjoint version of (2.14) is then given by:

$$\begin{aligned} v_t m + \frac{1}{2}(Av_y m)_y + Vvm = 0 \qquad & (t,y)\in(0,T)\times E \\ v(T,y) = 1 \qquad & y \in E \end{aligned} \tag{2.20}$$

The classical strategy is to solve such an equation by separation of variables: define the operator $-M$ with a certain domain $\mathcal{D}(-M) \subset L^2(E,m)$ as:

$$M\phi = \frac{(a^2\dot{\phi}m)^{\cdot}}{2m} + V\phi \qquad \phi \in \mathcal{D}(-M) \tag{2.21}$$

If $(-M, \mathcal{D}(-M))$ is Hilbert-Schmidt[3] then a natural guess for the solution to the boundary value problem is:

$$v(t,y) = \sum_{n\geq 0} \alpha_n \phi_n(y) e^{-\lambda_n(T-t)} \tag{2.22}$$

provided that $1 \in L^2(E,m)$ (or that m is a probability density), because v must satisfy the terminal condition $v(T,y) = 1$. Then, the coefficients $(\alpha_n)_{n\geq 0}$ are $\alpha_n = \int_E \phi_n(x)m(x)dx$. Crucially, α_n does not depend on T.

The expansion in (2.22) has important consequences for portfolio choice. It implies that v has the representation:

$$v(t,y) = e^{-\lambda_0(T-t)} \left(\alpha_0\phi_0(y) + \sum_{n\geq 1} \alpha_n \phi_n(y) e^{-(\lambda_n-\lambda_0)(T-t)} \right) \tag{2.23}$$

Consider the terms within the parentheses in the above expression. Since each eigenfunction $\phi_n, n \geq 1$ carries a weight $\alpha_n e^{-(\lambda_n-\lambda_0)(T-t)}$ that decreases in the horizon T, for long horizons both the value function u and the portfolio π are determined by the principal eigenfunction ϕ_0 alone. Indeed, (2.23) formally yields:

$$\lim_{T\uparrow\infty} \frac{v_y(t,y)}{v(t,y)} = \lim_{T\uparrow\infty} \frac{\alpha_0\dot{\phi}_0(y) + \sum_{n\geq 1} \alpha_n \dot{\phi}_n(y) e^{-(\lambda_n-\lambda_0)(T-t)}}{\alpha_0\phi_0(y) + \sum_{n\geq 1} \alpha_n \phi_n(y) e^{-(\lambda_n-\lambda_0)(T-t)}} = \frac{\dot{\phi}_0(y)}{\phi_0(y)} \tag{2.24}$$

provided that $\alpha_0 \neq 0$. When $\phi_0 > 0$, $\alpha_0 = \int_E \phi_0(x)m(x)dx > 0$. Regarding the optimal portfolios, the above calculation implies, by (2.15):

$$\lim_{T\uparrow\infty} \pi(t,y) = \frac{1}{1-p}\Sigma^{-1}\mu + \frac{\delta}{1-p}\Sigma^{-1}\Upsilon\frac{\dot{\phi}_0(y)}{\phi_0(y)} \tag{2.25}$$

Thus, as the horizon increases, the optimal portfolio converges to a time-homogeneous portfolio, which depends only on the current state variable y.

2.4 Static Fund Separation

Equation (2.25) paves the way to *static* fund separation for long term investments. Optimal portfolios for long planning horizons decompose as combinations of a fixed number of funds, such that:

i) The composition of each fund may depend on the state variable y, but is independent of the risk aversion $1-p$;

ii) Each investor chooses fund weights that depend on risk aversion $1-p$, but not on the state variable y.

[3] An operator is Hilbert-Schmidt if it is self-adjoint, with a discrete spectrum, bounded from below, tending to ∞, and the eigenfunctions $(\phi_n)_{n\geq 0}$ form an orthonormal basis for $L^2(E,m)$.

In practice, in models of interest it is possible to obtain a decomposition of the form:

$$\frac{\dot{\phi}_0(y)}{\phi_0(y)} = \sum_{i=1}^{m} w_i(p)\psi_i(y) \tag{2.26}$$

which implies that any optimal portfolio for a long horizon is the combination of $m+2$ funds, which are independent of the preference parameter p: the safe asset, the myopic portfolio $\Sigma^{-1}\mu$, and the m hedging portfolios $\Sigma^{-1}\Upsilon\psi_i(y)$. Risk aversion determines optimal portfolios only through their weights on these funds, but does not affect the funds themselves. Thus, the funds are the same for all long-horizon investors, regardless of their risk aversion.

Static fund separation entails a clear division of labor between an investor, or her financial planner, on one side, and an intermediary, such as a mutual fund manager, on the other. The investor or her financial planner choose the weights in the various funds, as they are in the best position to asses the investor's tolerance for risk. As in usual dynamic fund separation, investors do not need to trade the single securities in the funds' portfolios.

Furthermore – and this is the hallmark of *static* fund separation – investors do not trade in response to changes in the state variable y, but only rebalance as to keep fund proportions constant over time. Investors do not even need to *observe* the state variable. The manager, on the other hand, trades on behalf of several investors, with a broad range of risk attitudes. He does not need to know the risk tolerance of investors, because the composition of each fund is independent of preferences. On the contrary, the manager must observe the state variable, as this is the only trading signal affecting security weights within each fund.

The next sections carry out this program in detail for two classes of models, which nest several examples in the literature. For each class, the state variable follows one of the two basic stationary processes in Finance: the Ornstein-Uhlenbeck and the Feller diffusions. In both cases, it is possible to turn the above heuristic arguments into precise results, but at the cost of some careful parametric restrictions, which guarantee the well-posedness of optimization problems. In this regard, we tend to favor simpler to sharper results, sometimes concentrating on the most relevant case of higher risk aversion than logarithmic utility ($p < 0$).

There are two main technical difficulties in turning the heuristic arguments into precise statements. First, although finite linear combinations of functions of the form $e^{-\lambda_n(T-t)}\phi_n(y)$ satisfy the HJB equation, infinite sums may not commute with derivatives, especially when the state space E is not compact, as in the models considered. Checking that the guess in (2.22) solves (2.15) involves some careful arguments, which exploit the specific properties of the eigenfunctions ϕ_n in each model. Second, the solution to the HJB equation must correspond to the value function of the utility maximization problem – a verification theorem is needed.

3 Linear Model

This section studies a model with a state variable, which follows an Ornstein-Uhlenbeck process. The expected returns of the risky assets depend linearly on the state variable, while the interest rate and the volatilities are constant:

$$\begin{aligned} dR_t &= (\sigma\nu_0 + b\sigma\nu_1 Y_t)\, dt + \sigma dZ_t \\ dY_t &= -\, bY_t dt + dW_t \\ d\langle R, Y\rangle_t &= \rho dt \\ r(Y_t) &= r_0 \end{aligned} \tag{3.1}$$

where $\sigma \in \mathbb{R}^{n\times n}; \nu_0, \nu_1, \rho \in \mathbb{R}^n; b, r_0 > 0, \nu_1'\nu_1 > 0$. We consider the parametric restriction:

Assumption 3.1.

$$1 + q\rho'\nu_1 > 0 \tag{3.2}$$

To understand how this restriction arises, observe that m in (2.19) takes the form

$$m(y) = Ke^{-b(1+q\rho'\nu_1)y^2 - 2q\rho'\nu_0 y} \qquad y \in \mathbb{R} \tag{3.3}$$

where $K > 0$ is an arbitrary constant. Thus, unless (3.2) holds, there is no solution m with finite integral. The eigenvalue equation $-M\phi = \lambda\phi$ for M in (2.21) specifies to:

$$\frac{1}{2}\left(\dot{\phi}m\right)^{\cdot} + \left(\frac{1}{\delta}\left(pr_0 - \frac{q}{2}\nu_0'\nu_0 - qb\nu_0'\nu_1 y - \frac{q}{2}b^2\nu_1'\nu_1 y^2\right) + \lambda\right)\phi m = 0 \tag{3.4}$$

The following lemma identifies the solutions of (3.4) with those of the differential equation of the harmonic oscillator, under the additional parameter restriction:

Assumption 3.2.

$$b^2\left(\left(1 + q\rho'\nu_1\right)^2 + \frac{q}{\delta}\nu_1'\nu_1\right) > 0 \tag{3.5}$$

Remark 3.3. Note that (3.5) always holds if $p < 0$. For $0 < p < 1$, setting $\nu_1 = -\frac{1-\varepsilon}{q\rho'\rho}\rho$ for a small enough $\varepsilon > 0$ will cause (3.5) to fail even if Assumption 3.1 holds.

Lemma 3.4. *Let Assumptions 3.1 and 3.2. hold. Let $\phi \in L^2(\mathbb{R}, m)$, and define ψ by the equality:*

$$\phi(y) = \sqrt{\alpha}m(y)^{-1/2}\psi\left(\alpha y + \beta\right) \tag{3.6}$$

where the constants $\alpha > 0, \beta$, $\eta = K_1\lambda + K_2$ are in (C.1). *Then, ϕ solves* (3.4) *if and only if $\psi \in L^2(\mathbb{R})$ solves the ODE*

$$-\ddot{\psi}(z) + z^2\psi(z) = \eta\psi(z) \tag{3.7}$$

For $n = 0, 1, 2, \ldots$ consider the *Hermite functions*

$$\psi_n(z) = \sqrt{\frac{1}{n!2^n\sqrt{\pi}}}e^{-\frac{1}{2}z^2}h_n(z) \tag{3.8}$$

where h_n is the n^{th} Hermite polynomial defined by the recurrence relation

$$h_0(z) = 1; \qquad h_{n+1}(z) = 2zh_n(z) - \dot{h}_n(z), n = 0, 1, 2, \ldots$$

It is well known (Miklavčič, 1998, Section 2.9) that

$$\begin{aligned} &\psi_n \text{ solves (3.7) with } \eta_n = 2n + 1; n = 0, 1, 2, \ldots \\ &\left(\psi_n\right)_{n\in\mathbb{N}_0} \text{ is a complete orthonormal basis of } L^2(\mathbb{R}) \end{aligned} \tag{3.9}$$

Set

$$\varphi(z) = \left(\frac{1}{\alpha}m\left(\frac{z-\beta}{\alpha}\right)\right)^{1/2} \tag{3.10}$$

Assumption 3.1 implies that $\varphi \in L^2(\mathbb{R})$, hence the series

$$\sum_{n=0}^{M}\alpha_n\psi_n; \qquad \alpha_n = \int_{\mathbb{R}}\varphi(z)\psi_n(z)dz = \int_{\mathbb{R}}\phi_n(y)m(y)dy \tag{3.11}$$

Fund Name	Portfolio	Fund Weight
Myopic	$\Sigma^{-1}\mu(y)$	$w_\nu(p) = \frac{1}{1-p}$
Hedging Constant	$\Sigma^{-1}\Upsilon$	$w_{hc}(p) = \frac{\delta}{1-p} w_1(p)$
Hedging Linear	$\Sigma^{-1}\Upsilon y$	$w_{hl}(p) = \frac{\delta}{1-p} w_2(p)$

Table 3.1: Static Funds in the Linear Model. $w_1(p)$ and $w_2(p)$ are given in (3.15).

converges to φ in $L^2(\mathbb{R})$ as $M \uparrow \infty$. By construction of φ and (3.11) the series $\sum_{n=0}^{M} \alpha_n \phi_n$ converges to 1 in $L^2(\mathbb{R}, m)$ as $M \uparrow \infty$ and hence (2.22) is a candidate solution to (2.18).

The series in (3.11) converges to φ in a very strong sense: since φ is in the space of functions of rapid decrease, $\sum_n \alpha_n \psi_n$ converges to φ in this space (Reed and Simon (1972, Theorem V.13, Page 143)). In other words, for all non-negative integers l, m

$$\lim_{M\uparrow\infty} \sup_{z\in\mathbb{R}} \left| z^l \frac{d^m}{dz^m} \left(\sum_{n=0}^{M} \alpha_n \psi_n(z) - \varphi(z) \right) \right| = 0 \tag{3.12}$$

This convergence entails that the series expansion for v solves the PDE in (2.18). In summary, the following result holds:

Theorem 3.5. *Let Assumptions 3.1 and 3.2 hold. Define ψ_n, η_n as in* (3.9)*, λ_n, ϕ_n as in Lemma 3.4, and α_n as in* (3.11)*. Then*

i) the function

$$v(t, y) = \sum_{n=0}^{\infty} e^{-\lambda_n (T-t)} \alpha_n \phi_n(y) \tag{3.13}$$

is a strictly positive $C^{1,2}((0,T) \times E)$ solution of the partial differential equation in (2.18)*;*

ii) v satisfies the convergence property:

$$\lim_{T\uparrow\infty} \frac{v_y(t, y)}{v(t, y)} = \frac{\dot{\phi}_0(y)}{\phi_0(y)} \qquad \text{for all } t > 0, y \in \mathbb{R} \tag{3.14}$$

iii) the value function of the utility maximization problem (2.9) *is equal to $V(x, t, y) = \frac{x^p}{p} v(t, y)^\delta$;*

iv) the decomposition (2.26) *(static fund separation) holds with $m = 2$, $\psi_1(y) = 1$ and $\psi_2(y) = y$, and with the corresponding weights:*

$$\begin{aligned} w_1(p) &= q\rho'\nu_0 \left(1 - \frac{1}{\sqrt{1 + \frac{q\nu_1'\nu_1}{\delta(1+q\rho'\nu_1)^2}}} \right) - \frac{q\nu_0'\nu_1}{\delta(1 + q\rho'\nu_1)\sqrt{1 + \frac{q\nu_1'\nu_1}{\delta(1+q\rho'\nu_1)^2}}} \\ w_2(p) &= b(1 + q\rho'\nu_1) \left(1 - \sqrt{1 + \frac{q\nu_1'\nu_1}{\delta(1 + q\rho'\nu_1)^2}} \right) \end{aligned} \tag{3.15}$$

Wachter (2002) considers the following model with a single risky asset, in which the Sharpe ratio follows a mean reverting OU process:

$$\begin{aligned} dR_t &= \sigma X_t dt + \sigma dZ_t \\ dX_t &= b\left(\bar{X} - X_t\right) dt + \sigma_X dW_t \\ d\langle Z, W\rangle_t &= \rho dt \\ r(X_t) &= r_0 \end{aligned} \tag{3.16}$$

Parameter	Value
σ_X	0.0189
$\bar{X}$	0.0788
σ	0.0436
b	0.0226
ρ	-0.935
r_0	0.0014
ν_0	0.0788
ν_1	0.8363

Table 3.2: Parameter values for the linear model, as in Barberis (2000) and Wachter (2002). Time is in monthly units.

The transformation $Y = (X - \bar{X})/\sigma_X$ yields the model in (3.1) with $\nu_0 = \bar{X}$ and $\nu_1 = \sigma_X/b$.

For the parameter values in Table 3.2, Assumptions 3.1, 3.2 hold for risk aversion within the range $[.5, 10]$ used in the plots and tables below. Figure 1 shows the reduced value function v^δ as a function of the Sharpe ratio X as in (3.16), for various planning horizons, and for risk aversion of 10 ($p = -9$). Figure 2 plots the optimal portfolios as a function of the Sharpe ratio X for a number of horizons, including the myopic ($T \downarrow 0$) and long run ($T \uparrow \infty$) limits when the risk aversion is 10. Figure 2 directly compares to Figure 3 in Wachter (2002): the two figures are obtained from the same set of parameters, with the difference that Wachter (2002) approximates the correlation $\rho = -0.935$ with the value -1 required by her assumption of market completeness. Thus, our results show the effects of this approximation.

Figure 2 shows, compared to Wachter (2002), substantially lower stock holdings, especially at longer horizons. This finding is consistent with economic intuition: in a complete market, inter-temporal hedging is perfect, therefore the investor takes a larger hedging position. By contrast, in an incomplete setting, inter-temporal hedging is imperfect, therefore there is a tradeoff between hedging more, and adding more idiosyncratic volatility. The net result is that an investor hedges less. Since in this model hedging is achieved with a positive stock position, hedging less entails reducing stock holdings.

Table 3.3 below gives the respective fund weights for various values of the risk aversion $1 - p$ using the parameter values in Table 3.2. Fund weights are calculated for the low (2.5%) and high (97.5%) quantiles of the state variable, which represents the Sharpe ratio. Comparing the second and the last column in the table shows how inter-temporal hedging changes the variability of stock holdings across risk aversions, and for various values of the Sharpe ratio. For risk aversion greater than one, the difference between stock holdings in the high and low state is bigger in the last column: inter-temporal hedging leads to a larger variation between stock holdings in good and bad states, relative to the oscillation implied by the myopic strategy in the second column. The effect is reversed for risk aversion less than one, since such investors have negative hedging demands.

Another pattern emerges from the table: for the typical risk aversion greater than one, the hedging component is large when the Sharpe ratio is high, while it is close to zero when the Sharpe ratio is low. In other words, the optimal portfolio is substantially different than myopic when current investment opportunities are good, but it is very close to myopic when they are poor.

In summary, the hedging component amplifies the response of investors to changes in the state variable, and (for typical levels of risk aversion) becomes more visible when investment opportunities are good.

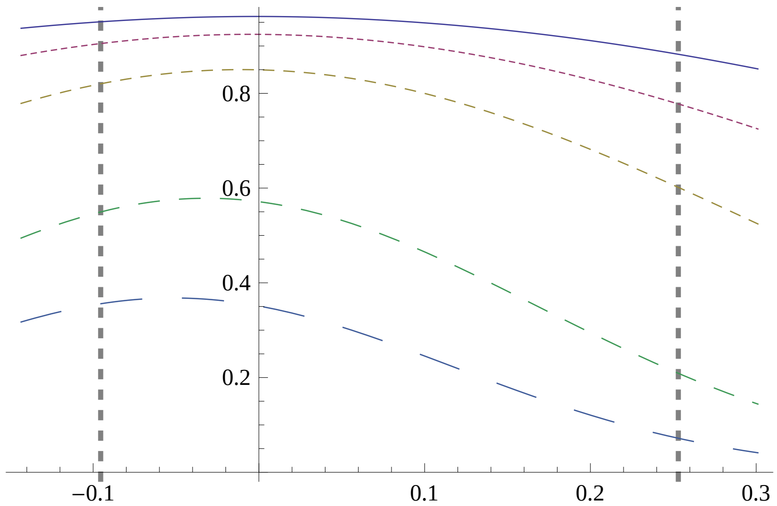

Figure 1: Reduced value function v^{δ} (y axis), in terms of the state variable (x axis), as the planning horizon varies from 3 months (solid), 6 months (tiny dashed), 1 year (short dashed), 3 years (medium dashed) and 5 years (long dashed). The state variable is the Sharpe ratio X from (3.16). Vertical lines are at the low (2.5%) and high (97.5%) percentiles of the state variable, under its stationary distribution. Risk aversion is 10 ($p = -9$), and market parameters are as in Table 3.2. Each plot uses 15 eigenfunctions from the series representation.

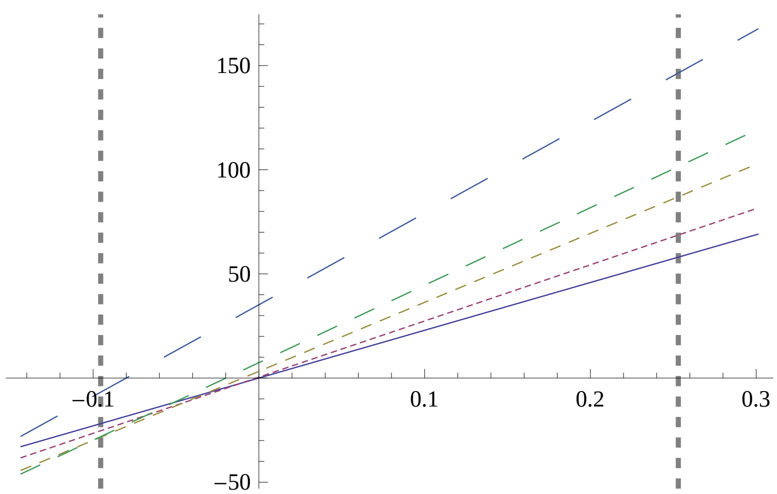

Figure 2: Optimal portfolio weights in the risky asset (y axis, in percent), in terms of the state variable (x axis), as the planning horizon varies from 0 (solid), 12 months (tiny dashed), 3 years (short dashed), 5 years (medium dashed) and ∞ (long dashed). The state variable is the Sharpe ratio X from (3.16). Vertical lines are at the low (2.5%) and high (97.5%) percentiles of the state variable, under its stationary distribution. Risk aversion is 10 ($p = -9$), and market parameters are as in Table 3.2. Each plot uses 15 eigenfunctions from the series representation.

Risk Aversion $1-p$	Myopic Fund Weight $w_\nu(p)$	Hedging Constant Fund Weight $w_{hc}(p)$	Hedging Linear Fund Weight $w_{hl}(p)$	
Risky (Low) Positions (High)	$w_\nu(p)\Sigma^{-1}\mu(y_{min})$ $w_\nu(p)\Sigma^{-1}\mu(y_{max})$	$w_{hc}(p)\Sigma^{-1}\Upsilon$ $w_{hc}(p)\Sigma^{-1}\Upsilon$	$w_{hl}(p)\Sigma^{-1}\Upsilon y_{min}$ $w_{hl}(p)\Sigma^{-1}\Upsilon y_{max}$	Totals
0.5	2.	0.073	0.01	
	-4.4	-1.6	2.	-3.9
	12.	-1.6	-2.	8.1
1.	1.	0.	0.	
	-2.2	0.	0.	-2.2
	5.8	0.	0.	5.8
2.	0.5	-0.028	-0.0029	
	-1.1	0.6	-0.57	-1.1
	2.9	0.6	0.57	4.1
3.	0.33	-0.033	-0.0031	
	-0.73	0.71	-0.61	-0.64
	1.9	0.71	0.61	3.3
5.	0.2	-0.032	-0.0027	
	-0.44	0.68	-0.54	-0.29
	1.2	0.68	0.54	2.4
10.	0.1	-0.024	-0.0019	
	-0.22	0.52	-0.37	-0.068
	0.58	0.52	0.37	1.5

Table 3.3: Fund weights and implied risky positions for the optimal long run portfolios, for different risk aversion levels (.5, 1, 2, 3, 5, 10). In each subpanel, the first row contain the static fund weights, while the second and third rows report the corresponding positions in the risky asset, respectively at the low (2.5%) and high (97.5%) quantiles of the state variable, under its stationary distribution. Market parameters are as in Table 3.2

4 Square Root Diffusion

In this model (cf. Guasoni and Robertson (2010)) a single state variable follows the square-root diffusion of Feller (1951), and simultaneously affects the interest rate, the volatilities of risky assets, and the Sharpe ratios.

$$\begin{aligned} dR_t &= (\sigma\nu_0 + \sigma\nu_1 Y_t)\, dt + \sqrt{Y_t}\sigma dZ_t \\ dY_t &= b(\theta - Y_t)\, dt + a\sqrt{Y_t} dW_t \\ d\langle R, Y\rangle_t &= \rho dt \\ r(Y_t) &= r_0 + r_1 Y_t \end{aligned} \tag{4.1}$$

Here $\sigma \in \mathbb{R}^{n\times n}$; $\nu_0, \nu_1, \rho \in \mathbb{R}^n$ and b, θ, a, r_0, r_1 are all positive reals.

Assumption 4.1.

$$b\theta - \frac{1}{2}a^2 > 0; \qquad b\theta - qa\rho'\nu_0 > 0; \qquad b + qa\rho'\nu_1 > 0 \tag{4.2}$$

These parametric restrictions deserve some comment. $b\theta > \frac{1}{2}a^2$ ensures that Y remains strictly positive at all times, so that $E = (0, \infty)$ To understand (4.2), note that m in (2.19) takes the form

$$m(y) = Ky^{\frac{2(b\theta - qa\rho'\nu_0)}{a^2} - 1} e^{-\frac{2(b+qa\rho'\nu_1)}{a^2}y} \qquad y > 0 \tag{4.3}$$

where $K > 0$ is an arbitrary constant. Thus, unless the second and third inequalities in (4.2) hold, there is no solution m with finite integral. The eigenvalue equation $-M\phi = \lambda\phi$ for M from (2.21) specifies to

$$\frac{1}{2}\left(a^2 y\dot{\phi} m\right)^{\cdot} + \left(\frac{1}{\delta}\left(-\frac{1}{2}q\nu_0'\nu_0\frac{1}{y} + (pr_0 - q\nu_0'\nu_1) + \left(pr_1 - \frac{1}{2}q\nu_1'\nu_1\right)y\right) + \lambda\right)\phi m = 0 \tag{4.4}$$

The following lemma identifies solutions of (4.4) with solutions of the generalized Laguerre differential equation under the additional parameter restriction:

Assumption 4.2.

$$p < 0 \tag{4.5}$$

Remark 4.3. For $0 < p < 1$ the results of this section still hold, but under very delicate parameter restrictions (similar, but more involved, than those in (3.5) from Assumption 3.2). Since the proofs would stay the same, the simpler case of $p < 0$ is considered here for clarity.

Lemma 4.4. *Let Assumptions 4.1 and 4.2 hold. Let $\phi \in L^2((0,\infty), m)$, and define ψ by the equality:*

$$\phi(y)m(y)^{1/2} = \sqrt{\zeta}\psi(\zeta y)\hat{m}(\zeta y)^{1/2} \tag{4.6}$$

where

$$\hat{m}(dz) = \frac{1}{\Gamma(\omega+1)} z^\omega e^{-z} dz \tag{4.7}$$

and the constants $\zeta > 0, \omega > 0$, $K_1 > 0$, K_2 and $\eta = K_1\lambda + K_2$ are defined in (D.1)*. Then ϕ solves* (4.4) *if and only if $\psi \in L^2((0,\infty), \hat{m})$ solves the ODE*

$$-z\ddot{\psi}(z) - (1 + \omega - z)\,\dot{\psi}(z) = \eta\psi(z) \tag{4.8}$$

For $n = 0, 1, 2, \ldots$ consider the *generalized Laguerre polynomial*

$$\psi_n(z) = \sqrt{\frac{\Gamma(\omega+1)}{n!\Gamma(n+\omega+1)}} z^{-\omega} e^z \frac{d^n}{dz^n} \left(e^{-z} z^{n+\omega} \right) \tag{4.9}$$

It is well known (Magnus et al., 1966, Ch 5.5) that

$$\begin{aligned} &\psi_n \text{ solves (4.8) with } \eta_n = n; n = 0, 1, 2, \ldots \\ &(\psi_n)_{n\in\mathbb{N}_0} \text{ is an orthonormal set in } L^2((0,\infty), \hat{m}) \end{aligned} \tag{4.10}$$

Set

$$\varphi(z) = \left(\frac{1}{\zeta} \frac{m(z/\zeta)}{\hat{m}(z)} \right)^{1/2} \tag{4.11}$$

By Assumption 4.1 $\varphi \in L^2((0,\infty), \hat{m})$. Since φ is evidently smooth, it follows from Lebedev (1972, Ch 5, Theorem 3) that the series

$$\sum_{n=0}^{M} \alpha_n \psi_n; \qquad \alpha_n = \int_0^\infty \varphi(z)\psi_n(z)\hat{m}(z)dz = \int_0^\infty \phi_n(y)m(y)dy \tag{4.12}$$

converges to φ pointwise (i.e. for all $z > 0$) and in $L^2((0,\infty), \hat{m})$ as $M \uparrow \infty$. By the construction of φ and (4.12) the series $\sum_{n=0}^{M} \alpha_n \phi_n$ converges to 1 pointwise and in $L^2((0,\infty), m)$ as $M \uparrow \infty$, hence it is a candidate solution to the PDE (2.18). Unlike the linear model in Section 3, the convergence does not take place in the strong sense of (3.12). However, asymptotic growth estimates for the Laguerre polynomials ψ_n imply the following convergence result:

Theorem 4.5. *Let Assumptions 4.1 and 4.2 hold. Define ψ_n as in* (4.9)*, η_n as in* (4.10)*, λ_n, ϕ_n as in Lemma 4.4, and α_n as in* (4.12)*. Then:*

i) the function

$$v(t,y) = \sum_{n=0}^{\infty} e^{-\lambda_n(T-t)} \alpha_n \phi_n(y) \tag{4.13}$$

is a strictly positive $C^{1,2}((0,T)\times(0,\infty))$ solution of the partial differential equation in (2.18)*;*

ii) v satisfies the convergence property:

$$\lim_{T\uparrow\infty} \frac{v_y(t,y)}{v(t,y)} = \frac{\dot{\phi}_0(y)}{\phi_0(y)} \qquad \textit{for all } t, y > 0 \tag{4.14}$$

iii) the value function of the utility maximization problem (2.9) *is equal to $V(x,t,y) = \frac{x^p}{p} v(t,y)^\delta$;*

iv) the decomposition (2.26) *(static fund separation) holds with $m = 2$, $\psi_1(y) = 1$ and $\psi_2(y) = 1/y$, and with the corresponding weights:*

$$\begin{aligned} w_1(p) &= \frac{1}{a^2} \left(b + qa\rho'\nu_1 - \sqrt{(b + qa\rho'\nu_1)^2 - \frac{2}{\delta} a^2 \left(pr_1 - \frac{1}{2} q\nu_1'\nu_1 \right)} \right) \\ w_2(p) &= \frac{1}{a^2} \left(\sqrt{\left(b\theta - qa\rho'\nu_0 - \frac{1}{2}a^2 \right)^2 + \frac{q}{\delta} a^2 \nu_0'\nu_0} - \left(b\theta - qa\rho'\nu_0 - \frac{1}{2}a^2 \right) \right) \end{aligned} \tag{4.15}$$

Fund Name	Portfolio	Fund Weight
Myopic	$\Sigma^{-1}\mu(y)$	$w_\nu(p) = \frac{1}{1-p}$
Hedging Constant	$\Sigma^{-1}\Upsilon$	$w_{hc}(p) = \frac{\delta}{1-p}w_1(p)$
Hedging Harmonic	$\Sigma^{-1}\Upsilon\frac{1}{y}$	$w_{hh}(p) = \frac{\delta}{1-p}w_2(p)$

Table 4.1: Static Funds in the Square Root Model. $w_1(p)$ and $w_2(p)$ are given in (4.15).

Parameter	Value
ν_0	0
ν_1	8.6
σ	1.00
b	8.1
θ	0.0137
a	0.32
ρ	-0.53
r_0	0.058
r_1	0

Table 4.2: Parameter values for the square root model.

Remark 4.6. For $\nu_0 = 0$, the above model is affine, and the parameter restrictions in Assumption 4.2 hold under Assumption 4.1 if $p < 0$. Furthermore, $w_2(p) = 0$, so that static separation holds with three funds rather than four.

For the parameter values in Table 4.2, each of Assumptions 4.1, 4.2 hold for risk aversion within the range $[1, 10]$ used in the plots and tables below. Figure 3 shows the reduced value function v^δ as a function of the state variable Y for various planning horizons and for risk aversion of 10 ($p = -9$). Figure 4 plots the optimal portfolios as a function of the state variable Y for a number of horizons, including the myopic ($T \downarrow 0$) and long run ($T \uparrow \infty$) limits when the risk aversion is 10. Figure 4 shows how the optimal portfolio shifts from the myopic to the long run limit, as the horizon increases. Each plot is generated using 15 eigenfunctions.

Table 4.3 below gives the respective fund weights for various values of the risk aversion $1 - p$ using the parameter values in (4.2). Fund weights are given for high and low percentiles of the state variable.

5 Conclusion

If static fund separation holds, optimal portfolios for long term CRRA investors are constant mixes of a few common funds. Then, hedging demands are obtained explicitly as combinations of positions in one or more hedging funds, which depend on the model, and may be larger than the number of state variables. Merton (1973)'s dynamic fund separation implies that hedging funds are (locally) perfectly correlated with each other, but their covariance changes over time, as it depends on the state variable.

This paper establishes static fund separation for two common classes of models, in which the state variable is either an Ornstein-Uhlenbeck or a Feller diffusion. A central technique is the eigenvalue decomposition of value functions made possible by the linear HJB equation, which follows from constant asset-state correlations, and a single state variable. Static fund separation in models with several state variables remains an open area of research.

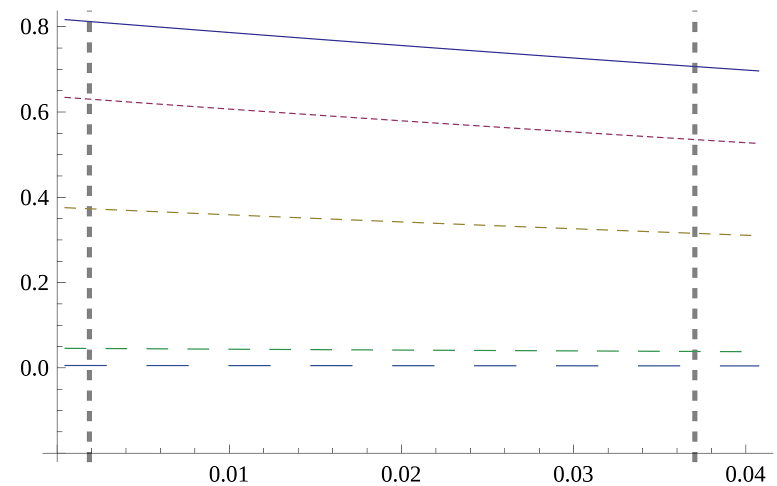

Figure 3: The plot displays the reduced value function v^{δ} (y axis), in terms of the state variable (x axis), as the planning horizon varies from 3 months (solid), 6 months (tiny dashed), 1 year (short dashed), 3 years (medium dashed) and 5 years (long dashed). Vertical lines are at the low (2.5%) and high (97.5%) percentiles of the state variable, under its stationary distribution. Risk aversion is 10 ($p = -9$), and market parameters are as in Table 4.2. Each plot uses 15 eigenfunctions from the series representation.

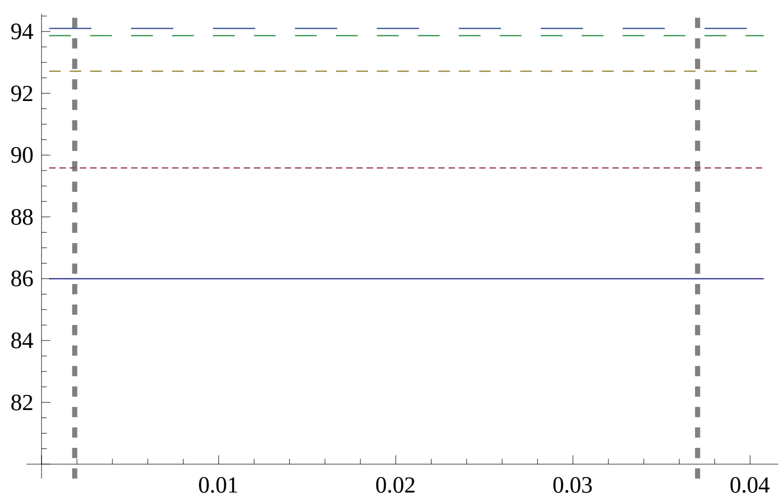

Figure 4: The plot displays the optimal portfolio weights in the risky asset (y axis), in terms of the state variable (x axis), as the planning horizon varies from $T \downarrow 0$ (solid), 1 month (tiny dashed), 3 months (short dashed), 6 months (medium dashed) and $T \uparrow \infty$ (long dashed). Portfolio weights are given in terms of percentages. Vertical lines are at the low (2.5%) and high (97.5%) percentiles of the state variable, under its stationary distribution. Risk aversion is 10 ($p = -9$), and market parameters are as in Table 4.2. Each plot uses 15 eigenfunctions from the series representation.

Risk Aversion $1-p$ Risky Positions	Myopic Fund Weight $w_\nu(p)$ $w_\nu(p)\Sigma^{-1}\mu$	Hedging Constant Fund Weight $w_{hc}(p)$ $w_{hc}(p)\Sigma^{-1}\Upsilon$	Totals
1.	1.	0.	
	8.6	0.	8.6
2.	0.5	-1.2	
	4.3	0.21	4.5
3.	0.33	-1.1	
	2.9	0.19	3.1
5.	0.2	-0.83	
	1.7	0.14	1.9
10.	0.1	-0.48	
	0.86	0.081	0.94

Table 4.3: Fund weights and implied risky positions for the optimal long run portfolios, for different risk aversion levels (1, 2, 3, 5, 10). In each subpanel, the first row contains the static fund weights, while the second row reports the corresponding positions in the risky asset. Market parameters are as in Table 4.2

A Feynman-Kač Extension

Let $E = (\alpha, \beta)\,, -\infty \leq \alpha < \beta \leq \infty$ be an open interval in $\mathbb{R}$, and $C^{m,\alpha}(E, \mathbb{R}^d)$ the class of $\mathbb{R}^d$-valued continuous maps on E whose m-th derivative is uniformly α-Hölder continuous on compact subsets of E. Set $C^m(E, \mathbb{R}^d) = C^{m,0}(E, \mathbb{R}^d)$. $\Omega^d = C([0, \infty), \mathbb{R}^d)$ denotes the space of continuous paths from $[0, \infty)$ to $\mathbb{R}^d$, endowed with the Borel σ-algebra $\mathcal{F}$. For notational simplicity $\Omega^1 = \Omega$.

The following extension of the Feynman-Kač formula is needed for the proofs of Theorem 3.5 and Theorem 4.5. Because Proposition A.2 will be used in a number of contexts the A and b in Assumption A.1 and Proposition A.2 are not necessarily the same A and b as in (2.2)–(2.3).

Assumption A.1. *Assume, for some $\gamma \in (0, 1]$, $A \in C^{2,\gamma}(E)\,; A(z) > 0, z \in E$ and $b \in C^{1,\gamma}(E, \mathbb{R})$. Assume there exists a solution $(\mathbb{Q}_z)_{z\in E}$ to the martingale problem for L on E where L is given by*

$$L = \frac{1}{2} A(z) \frac{d^2}{dz^2} + b(z) \frac{d}{dz}$$

Let Z denote the coordinate mapping process.

Proposition A.2. *Let Assumption A.1 hold. Let $f \in C^2(E); f(z) > 0, z \in E$ be such that for any $T > 0$ and $z \in E$*

$$\sup_{0 \leq t \leq T} E_z^{\mathbb{Q}} \left[f(Z_{T-t}) \right] < \infty \tag{A.1}$$

Then the function $h(t, z) : (0, T) \times E \mapsto (0, \infty)$ defined by

$$h(t, z) = E_z^{\mathbb{Q}} \left[f(Z_{T-t}) \right]$$

is in $C^{1,2}((0, T) \times E)$ and satisfies the following differential expression and terminal condition

$$\begin{aligned} \partial_t h(t, z) + Lh(t, z) = 0 \qquad & (t, z) \in (0, T) \times E \\ h(T, z) = f(z) \qquad & z \in E \end{aligned} \tag{A.2}$$

Furthermore $h(t, Z_t)/h(0, z)$ is a $\mathbb{Q}_z$ martingale for all $z \in E$.

Proof of Proposition A.2. Note that the classical version of the Feynman-Kac formula (Friedman, 1975, Theorem 5.3) does not apply directly because a) A is not bounded away from 0 on E, and b) f may grow faster than polynomial near the boundary of E. Rather, the statement is proved using Heath and Schweizer (2000, Theorem 1), which yields that h is a classical solution of (A.2).

To check that the assumptions of Theorem 1 in Heath and Schweizer (2000) are satisfied, note that, since A is locally Lipschitz on E due to Assumption A.1, Lemma 1.1 in Friedman (1975) pp. 128 implies that a is also locally Lipschitz on E where $a = \sqrt{A}$. On the other hand, the local Lipschitz continuity of b is also ensured by Assumption A.1. Hence (A1) in Heath and Schweizer (2000) is satisfied. Second, (A2) in Heath and Schweizer (2000) holds thanks to the well-posedness of the martingale problem $(\mathbb{Q}_z)_{z\in E}$, in particular the coordinate process Z does not hit the boundary of E under $\mathbb{Q}_x$. Third, (A3') in Heath and Schweizer (2000), (A3a')-(A3d') are clearly satisfied under our assumptions.

In order to check (A3e'), it suffices to show that h is continuous in any compact sub-domain of $(0, T) \times E$. To this end, recall that the domain is $E = (\alpha, \beta)$ for $-\infty \leq \alpha < \beta \leq \infty$. Let $\{\alpha_m\}$ and $\{\beta_m\}$ two sequences such that $\alpha_m < \beta_m$ for all m, α_m strictly decreases to α, and β_m strictly increases to β. Set $E_m = (\alpha_m, \beta_m)$. For each m there exists a function $\psi_m(z) \in C^\infty(E)$ such that

a) $\psi_m(z) \leq 1$, b) $\psi_m(z) = 1$ on E_m, and c) $\psi_m(z) = 0$ on $E \cap E_{m+1}^c$. To construct such ψ_m let $\varepsilon_m = \frac{1}{3} \min \{\beta_{m+1} - \beta_m, \alpha_m - \alpha_{m+1}\}$ and then take

$$\psi_m(z) = \eta_{\varepsilon_m} * 1_{\{\alpha_m - \varepsilon_m, \beta_m + \varepsilon_m\}}(z),$$

where η_{ε_m} is the standard mollifier and $*$ is the convolution operator. Define the functions f_m and h^m by

$$f_m(z) = \psi_m(z) f(z) \quad \text{and} \quad h^m(t, z) = E_z^{\mathbb{Q}} [f_m(Z_{T-t})].$$

By construction, for all $z \in E$, $\uparrow \lim_{m\uparrow\infty} f_m(z) = f(z)$. It then follows from the monotone convergence theorem and (A.1) that $\lim_{m\uparrow\infty} h^m(t, z) = h(t, z)$. Since $f \in C^2(E, (0, \infty))$ and $f > 0$, each $f_m(z) \in C^2(E, [0, \infty))$ is bounded. It then follows from the Feller property for $(\mathbb{Q}_z)_{z\in E}$ (Pinsky, 1995, Theorem 1.13.1) that h^m is continuous in z. On the other hand, by construction of f_m there exists a constant $K_m > 0$ such that

$$a|\dot{f}_m| \leq K_m, \qquad |Lf_m| \leq K_m, \quad \text{on } E. \tag{A.3}$$

Moreover, Ito's formula gives that, for any $0 \leq s \leq t \leq T$,

$$f_m(Z_t) = f_m(Z_s) + \int_s^t Lf_m(Z_u)du + \int_s^t a\dot{f}_m(Z_u)dW_u.$$

where W is a $\mathbb{Q}_z$ Brownian Motion. Combining the previous equation with estimates in (A.3), it follows that:

$$\sup_{z\in E} \left| E_z^{\mathbb{Q}} [f_m(Z_t) - f_m(Z_s)] \right| \leq K_m(t - s).$$

Therefore, h^m is uniformly continuous in t. Combining with the continuity of h^m in z, we conclude that h^m is jointly continuous in (t, z) on $[0, T] \times E$.

Note that the operator L is uniformly elliptic in the parabolic domain $(0, T) \times E_m$. It then follows from a straightforward calculation that h^m satisfies the differential equation:

$$\partial_t h^m + Lh^m = 0 \quad (t, z) \in (0, T) \times E_m.$$

Note that $(h^m)_{m\geq 0}$ is uniformly bounded from above by h, which is finite on $[0, T] \times E_m$. Appealing to the *interior Schauder estimate* (Friedman, 1964, Theorem 15) there exists a subsequence $(h^{m'})_{m'}$ which converges to h uniformly in $(0, T) \times D$ for any compact sub-domain D of E. Since each $h^{m'}$ is continuous and the convergence is uniform, we confirm that h is continuous in $(0, T) \times D$. Since the choice of D is arbitrary in E, (A3e') in Heath and Schweizer (2000) is satisfied. Therefore, h satisfies (A.2).

It is now shown that $h(t, Z_t)/h(0, z)$ is a $\mathbb{Q}_z$ martingale for all $z \in E$. The dynamics of X under $\mathbb{Q}_z$ are

$$dZ_t = b(Z_t)dt + a(Z_t)dW_t; \qquad Z_0 = z$$

where W is a $\mathbb{Q}_z$ Brownian motion. Using the differential expression for h in (A.2), Itô's formula implies for all $0 \leq t \leq T$ that

$$\frac{h(t, Z_t)}{h(0, z)} = \mathcal{E}\left(\int_0^{\cdot} a\frac{h_z}{h}(Z_s)dW_s\right)_t \tag{A.4}$$

It follows from the terminal condition in (A.2) that $0 < E_z^{\mathbb{Q}} [h(T, Z_T)] = h(0, z) < \infty$. Then the right hand side of (A.4) is a strictly positive $\mathbb{Q}_z$ martingale on $[0, T]$.

□

B Verification

The following assumptions are on the coefficients $\mu, b, \Sigma, A, \Upsilon$ given in (2.2)–(2.3). Note that these assumptions are satisfied by all the models considered within this paper. The first assumption requires the regularity and non-degeneracy of coefficients, while the second one guarantees that they identify the law of Y.

Assumption B.1. *There exists $\alpha \in (0,1]$ such that $r \in C(E,\mathbb{R})$, $b \in C^{1,\alpha}(E,\mathbb{R})$, $\Upsilon, \mu \in C^{1,\alpha}(E,\mathbb{R}^n)$, $\Sigma \in C^{1,\alpha}(E,\mathbb{R}^{n\times n})$ and $A \in C^{2,\alpha}(E,\mathbb{R})$. $A > 0$ and Σ is positive definite for all $y \in E$, uniformly on all compact subsets of E.*

Assumption B.2. *For all $y \in E$ there exists a unique probability P^y on $(\Omega^{n+1}, \mathcal{F})$ such that (R, Y) satisfies* (2.2)–(2.5) *with initial condition $(R_0, Y_0) = (0, y)$, and $P^y(Y_t \in E, \forall t \geq 0) = 1$. The family $(P^y)_{y\in E}$ has the strong Markov property.*

The value function of the utility maximization problem is defined as the expected utility, conditional on the current state $y \in E$, the time $t \in [0,T]$ and the current wealth $x > 0$:

$$V(t,x,y) = \sup_{(\pi_u)_{u\geq t}} \frac{1}{p} E\left[(X_T^\pi)^p \,|Y_t = y, X_t = x\right]$$

The following lemma establishes conditions under which a solution to the HJB equation is indeed the value function.

Lemma B.3. *Let Assumption B.1 hold. For b^ν, V as in* (2.16) *and L^ν as in* (2.17) *assume that:*

i) $v(t,y)$ is a strictly positive $C^{1,2}((0,T)\times E,\mathbb{R})$ solution to

$$\begin{aligned} v_t + L^\nu v + V v = 0 \qquad & (t,y) \in (0,T)\times E \\ v(T,y) = 1 \qquad & y \in E \end{aligned} \tag{B.1}$$

ii) both the original and the auxiliary models:

$$(P) \begin{cases} dR_t = \mu dt + \sigma dZ_t \\ dY_t = b dt + a dW_t \end{cases} \quad (\hat{P}_T) \begin{cases} dR_t = \frac{1}{1-p}\left(\mu + \delta\Upsilon\frac{v_y}{v}\right)dt + \sigma d\hat{Z}_t \\ dY_t = \left(b^\nu + A\frac{v_y}{v}\right)dt + a d\hat{W}_t \end{cases} \tag{B.2}$$

satisfy Assumption B.2 on $[0,T]$ under equivalent probabilities P^y and $\hat{P}_T^y$, $y \in E$.

Then $V(t,x,y) = \frac{x^p}{p}\left(v(t,y)\right)^\delta$. The optimal trading strategy is given by

$$\pi_t = \frac{1}{1-p}\Sigma^{-1}\left(\mu + \delta\Upsilon\frac{v_y}{v}\right) \tag{B.3}$$

evaluated at (t, Y_t).

Proof of Lemma B.3. For any function $\eta \in C((0,T)\times E,\mathbb{R})$ set

$$\begin{aligned} h(t,y) &= -\rho'\sigma^{-1}(y)\mu(y) + \left(1-\rho'\rho\right)a(y)\eta(t,y) \\ g(t,y) &= -\bar{\rho}'\sigma^{-1}(y)\mu(y) - \bar{\rho}'\rho a(y)\eta(t,y) \end{aligned} \tag{B.4}$$

where $\bar\rho$ is the unique non-negative definite square root of $1-\rho\rho'$. Note that $1-\rho\rho$ being non-negative definite is equivalent to $1-\rho'\rho\geq 0$. Using the decomposition $Z=\rho W+\bar\rho B$ where B is a Brownian Motion independent of W define the process

$$Z^\eta_t=\mathcal{E}\left(\int_0^\cdot h(s,Y_s)'dW_s+\int_0^\cdot g(s,Y_s)'dB_s\right)_t\qquad 0\leq t\leq T$$

The continuity of η and Assumptions B.1,B.2 ensure that Z^η is a strictly positive P^y local martingale on $[0,T]$ for any $T>0$ and all $y\in E$. Furthermore, using stochastic integration by parts, it follows that for any trading strategy π the process

$$e^{-\int_0^\cdot rds}Z^\eta_\cdot X^\pi_\cdot$$

is a non-negative P^y supermartingale. Set $M^\eta=e^{-\int_0^\cdot rds}Z^\eta$. Using the well known duality results relating terminal wealths from trading strategies and process which render trading strategies non-negative supermartingales (see Kramkov and Schachermayer (1999)) it suffices to show that under the assumption of well-posedness for $(\hat P^y_T)_{y\in E}$:

$$E\left[\frac{1}{p}(X^\pi_T)^p|X_t=x,Y_t=y\right]=\frac{x^p}{p}\left(E\left[\left(\frac{M^\eta_T}{M^\eta_t}\right)^q|Y_t=y\right]\right)^{1-p}$$

for v solving (B.1), π as in (B.3) and $\eta=\delta\frac{v_y}{v}$. Plugging in for X^π_T, M^η_T/M^η_t this is equivalent to showing

$$\begin{aligned}E&\left[\exp\left(p\int_t^T\left(r+\pi'\mu-\frac{1}{2}\pi'\Sigma\pi\right)ds+\int_t^T p\pi'\sigma\rho dW_s+\int_t^T p\pi'\sigma\bar\rho dB_s\right)|Y_t=y\right]\\&=\left(E\left[\exp\left(q\int_t^T\left(-r-\frac{1}{2}h'h-\frac{1}{2}g'g\right)ds+\int_t^T qh'dW_s+\int_t^T qg'dB_s\right)|Y_t=y\right]\right)^{1-p}\end{aligned}\tag{B.5}$$

Under Assumptions B.2, B.1, the smoothness of v and the well-posedness under P^y and $\hat P^y_T$ imply that $\hat Z_t=d\hat P_T/dP\big|_{\mathcal{F}_t}$ is a martingale (see Cheridito et al. (2005), Theorem 2.4). $\hat Z$ takes the form

$$\hat Z_t=\mathcal{E}\left(\int_0^\cdot\left(-q\Upsilon'\Sigma^{-1}\mu+A\frac{v_y}{v}\right)'a^{-1}dW_s-q\int_0^\cdot\left(\Sigma^{-1}\mu+\delta\Sigma^{-1}\Upsilon\frac{v_y}{v}\right)'\sigma\bar\rho dB_s\right)_t\tag{B.6}$$

It is convenient to define $w(t,y)=\delta\log v(t,y)$. w solves the semi-linear PDE

$$\begin{aligned}w_t+\frac{1}{2}Aw_{yy}+\frac{1}{2\delta}Aw_y^2+b^\nu w_y+\delta V&=0\qquad (t,y)\in(0,T)\times E\\ w(T,y)&=0\qquad y\in E\end{aligned}\tag{B.7}$$

With this notation, $\pi=\frac{1}{1-p}\Sigma^{-1}\left(\mu+\Upsilon w_y\right)$ and $\eta=w_y$. It is now claimed that, for $Y_t=y$:

$$\begin{aligned}\exp\left(p\int_t^T\left(r+\pi'\mu-\frac{1}{2}\pi'\Sigma\pi\right)ds+\int_t^T p\pi'\sigma\rho dW_s+\int_t^T p\pi'\sigma\bar\rho dB_s\right)&=\frac{\hat Z_T}{\hat Z_t}e^{w(t,y)}\\ \exp\left(-q\int_t^T\left(r+\frac{1}{2}h'h+\frac{1}{2}g'g\right)ds+\int_t^T qh'dW_s+\int_t^T qg'dB_s\right)&=\frac{\hat Z_T}{\hat Z_t}e^{\frac{1}{1-p}w(t,y)}\end{aligned}\tag{B.8}$$

from which (B.5) follows. Under P^y, using Ito's formula and (B.7) it follows that given $Y_t=y$

$$\begin{aligned}-w(t,y)&=\int_t^T w_tds+\int_t^T bw_yds+\int_t^T aw_ydW_s+\frac{1}{2}\int_t^T Aw_{yy}ds\\&=-\int_t^T\left(\frac{1}{2\delta}Aw_y^2-qw_y\Upsilon'\Sigma^{-1}\mu+pr-\frac{q}{2}\mu'\Sigma^{-1}\mu\right)ds+\int_t^T w_yadW_s\end{aligned}$$

plugging back in for b^ν and V. At this point, verifying (B.8) amounts to plugging in for $\hat{Z}_T/\hat{Z}_t$, π, η (and hence h, g via (B.4)) and w and then matching up the ds, dW and dB terms. Performing these steps readily yields the conclusion (recalling that $A/\delta = A - q\Upsilon'\Sigma^{-1}\Upsilon$). $\square$

C Proofs of Section 3

The following constants are used in Lemma 3.4 and Theorem 3.5:

$$\begin{aligned}
\mathbf{A} &= 2qb\left(\rho'\nu_0\left(1+q\rho'\nu_1\right)+\frac{1}{\delta}\nu_0'\nu_1\right)\\
\mathbf{B} &= b^2\left(\left(1+q\rho'\nu_1\right)^2+\frac{q}{\delta}\nu_1'\nu_1\right)\\
\mathbf{C} &= -q^2\left(\rho'\nu_0\right)^2+b\left(1+q\rho'\nu_1\right)-\frac{q}{\delta}\nu_0'\nu_0+\frac{2pr_0}{\delta}\\
\alpha &= \mathbf{B}^{1/4}; \qquad \beta = \frac{\mathbf{A}}{2\mathbf{B}^{3/4}}\\
K_1 &= \frac{2}{\sqrt{\mathbf{B}}}; \qquad K_2 = \frac{1}{\sqrt{\mathbf{B}}}\left(\frac{\mathbf{A}^2}{4\mathbf{B}}+\mathbf{C}\right)
\end{aligned} \tag{C.1}$$

Note that (3.5) in Assumption 3.2 is equivalent to $\mathbf{B} > 0$.

The proof of Lemma 3.4 and Proposition 3.5 require the following three lemmas:

Lemma C.1. *Let Assumptions 3.1 and 3.2 hold. Let m be as in* (3.3)*, α, β, K_1 and K_2 be as in* (C.1)*, $\lambda \in \mathbb{R}$ and $\eta = K_1\lambda + K_2$. Let*

$$V(y) = \frac{1}{\delta}\left(pr_0 - \frac{q}{2}\nu_0'\nu_0 - qb\nu_0'\nu_1 y - \frac{q}{2}b^2\nu_1'\nu_1 y^2\right) \tag{C.2}$$

If $f, g \in C^{1,2}((0,T)\times\mathbb{R})$ are related by

$$f(t,y) = \sqrt{\alpha}m(y)^{-1/2}g(t,z) \Longleftrightarrow f(t,y) = \varphi(z)^{-1}g(t,z)$$

for $z = \alpha y + \beta$ then at $z = \alpha y + \beta$

$$\begin{aligned}
&\frac{1}{2}\left(\dot{f}(t,y)m(y)\right)^{\cdot} + V(y)f(t,y)m(y) + \lambda f(t,y)m(y) + f_t(t,y)m(y)\\
&\qquad = -\frac{1}{2}\alpha^{5/2}m\left(\frac{z-\beta}{\alpha}\right)^{1/2}\left(-\ddot{g}(z) + z^2 g(z) - \eta g(z) - \frac{2}{\alpha^2}g_t(t,z)\right)
\end{aligned} \tag{C.3}$$

Proof of Lemma C.1. That $f_t(t,y)m(y) = \sqrt{\alpha}m((z-\beta)/\alpha)^{1/2}g_t(t,z)$ is immediate. For the remaining terms, it suffices to think of f, g as functions of y (resp z) alone. Define $\Xi(y)$ by

$$\Xi(y) = \frac{1}{2}\left(\dot{f}(y)m(y)\right)^{\cdot} + V(y)f(y)m(y) + \lambda f(y)m(y) \tag{C.4}$$

Let $h(y) = m^{1/2}(y)f(y)$. It follows that

$$\begin{aligned}
\dot{f} &= m^{-1/2}\left(\dot{h} - \frac{\dot{m}}{2m}h\right)\\
\ddot{f} &= m^{-1/2}\left(\ddot{h} - 2\frac{\dot{m}}{2m}\dot{h} + \left(\left(\frac{\dot{m}}{2m}\right)^2 - \left(\frac{\dot{m}}{2m}\right)^{\cdot}\right)h\right)
\end{aligned}$$

Therefore, from (C.4) it follows that:

$$\begin{aligned}
\Xi &= \frac{1}{2}\left(\dot{f}m\right)^{\cdot} + (V+\lambda)fm \\
&= m^{1/2}\left(\frac{1}{2}\ddot{h} - \frac{\dot{m}}{2m}\dot{h} + \frac{1}{2}\left(\left(\frac{\dot{m}}{2m}\right)^2 - \left(\frac{\dot{m}}{2m}\right)^{\cdot}\right)h + \frac{\dot{m}}{2m}\dot{h} - \left(\frac{\dot{m}}{2m}\right)^2 h + (V+\lambda)h\right) \\
&= m^{1/2}\left(\frac{1}{2}\ddot{h} + \left(V+\lambda - \frac{1}{2}\left(\frac{\dot{m}}{2m}\right)^2 - \frac{1}{2}\left(\frac{\dot{m}}{2m}\right)^{\cdot}\right)h\right)
\end{aligned}$$

Plugging in for V from (C.2) and in for $\frac{\dot{m}}{m}$ from (3.3) and multiplying by -2 yields

$$-2\Xi = m^{1/2}\left(-\ddot{h} + \left(\mathbf{B}y^2 + \mathbf{A}y - \mathbf{C} - 2\lambda\right)h\right) \tag{C.5}$$

for $\mathbf{A}, \mathbf{B}, \mathbf{C}$ as in (C.1). An affine transformation in the state variable normalizes the quadratic constant $\mathbf{B}$ and eliminates the linear constant $\mathbf{A}$. Let $z = \alpha y + \beta$. By construction, $g(z) = \frac{1}{\sqrt{\alpha}}h(y) = \frac{1}{\sqrt{\alpha}}h(\frac{z-\beta}{\alpha})$. Evaluating (C.5) at $y = \frac{z-\beta}{\alpha}$ yields

$$\begin{aligned}
&-2\Xi\left(\frac{z-\beta}{\alpha}\right) \\
&= \alpha^{5/2}m\left(\frac{z-\beta}{\alpha}\right)^{1/2}\left(-\ddot{g}(z) + \left(\frac{\mathbf{B}}{\alpha^4}z^2 + \left(\frac{\mathbf{A}}{\alpha^3} - \frac{2\beta\mathbf{B}}{\alpha^4}\right)z + \left(\frac{\beta^2\mathbf{B}}{\alpha^4} - \frac{\beta\mathbf{A}}{\alpha^3} - \frac{\mathbf{C}}{\alpha^2} - \frac{2\lambda}{\alpha^2}\right)\right)g(z)\right)
\end{aligned} \tag{C.6}$$

Using the representations for $\alpha, \beta, \mathbf{A}, \mathbf{B}, \mathbf{C}, K_1$ and K_2 from (C.1) and η from the statement of the Lemma it follows that

$$-2\Xi\left(\frac{z-\beta}{\alpha}\right) = \alpha^{5/2}m\left(\frac{z-\beta}{\alpha}\right)^{1/2}\left(-\ddot{g}(z) + z^2 g(z) - \eta g(z)\right), \tag{C.7}$$

which is the desired result. □

Lemma C.2. *Let $(h_n)_{n\geq 1}$ be a sequence of real numbers and let $(\gamma_n)_{n\geq 1}$ be a decreasing sequence of positive real numbers. Then, for all $m, M \in \mathbb{N}$, $m \leq M$*

$$\left|\sum_{n=m}^{M}\gamma_n h_n\right| \leq \gamma_m \max_{N=m,\ldots M}\left|\sum_{n=m}^{N} h_n\right|$$

Proof of Lemma C.2. For the γ_n of the statement

$$\begin{aligned}
\left|\sum_{n=m}^{M}\gamma_n h_n\right| &= \left|\gamma_M\sum_{n=m}^{M} h_n + (\gamma_{M-1} - \gamma_M)\sum_{n=m}^{M-1} h_n + \ldots + (\gamma_m - \gamma_{m+1})h_m\right| \\
&\leq \gamma_M\left|\sum_{n=m}^{M} h_n\right| + (\gamma_{M-1} - \gamma_M)\left|\sum_{n=m}^{M-1} h_n\right| + \ldots + (\gamma_m - \gamma_{m+1})\left|h_m\right| \\
&\leq \max_{N=m,\ldots M}\left|\sum_{n=m}^{M} h_n\right|\left(\gamma_M + (\gamma_{M-1} - \gamma_M) + \ldots + (\gamma_m - \gamma_{m+1})\right) \\
&= \gamma_m \max_{N=m,\ldots M}\left|\sum_{n=m}^{M} h_n\right|
\end{aligned}$$

□

Proof of Lemma 3.4. By applying Lemma C.1 for $f(y) = \phi(y)$, $g(z) = \psi(z)$ (note there is no dependence upon t) it follows that ϕ satisfies (3.4) for some $\lambda \in \mathbb{R}$ if and only if ψ satisfies (3.7) for $\eta = K_1\lambda + K_2$. As for the respective L^2 norms, since

$$\int_{\mathbb{R}} \psi(z)^2 dz = \frac{1}{\alpha}\int_{\mathbb{R}} \phi\left(\frac{z-\beta}{\alpha}\right)^2 m\left(\frac{z-\beta}{\alpha}\right) dz = \int_{\mathbb{R}} \phi(y)^2 m(y) dy$$

$\psi \in L^2(\mathbb{R})$ if and only if $\phi \in L^2(\mathbb{R}, m)$ and they both have the same norm.

□

Proof of Theorem 3.5. It is first shown that v from (3.13) is a strictly positive $C^{1,2}((0,T)\times\mathbb{R})$ solution to the PDE in (2.18), or equivalently, to the PDE in (2.20), specified to the model in (3.1). We have, using (3.6), (3.9),(3.10), and (C.1) that for any $y \in \mathbb{R}$, at $z = \alpha y + \beta$

$$v(t,y) = \sum_{n=0}^{\infty} e^{-\lambda_n(T-t)}\alpha_n\phi_n(y) = e^{K_2/K_1(T-t)}\varphi(z)^{-1}\sum_{n=0}^{\infty} e^{-\eta_n/K_1(T-t)}\alpha_n\psi_n(z) \tag{C.8}$$

Set

$$w(t,z) = \sum_{n=0}^{\infty} e^{-\eta_n/K_1(T-t)}\alpha_n\psi_n(z) \tag{C.9}$$

so that (C.8) becomes

$$v(t,y) = e^{K_2/K_1(T-t)}\varphi(z)^{-1}w(t,z) \tag{C.10}$$

By applying Lemma C.1 for

$$\lambda = 0; \quad f(t,y) = v(t,y); \quad g(t,z) = w(t,z)e^{K_2/K_1(T-t)}$$

it follows that $v(t,y)$ solves (2.20) if and only if $w(t,z)$ solves (note $\alpha^2/2 = 1/K_1$)

$$\begin{aligned} K_1 w_t + w_{zz} - z^2 w = 0 \qquad & (t,z) \in (0,T)\times\mathbb{R} \\ w(T,z) = \varphi(z) \qquad & z \in \mathbb{R} \end{aligned} \tag{C.11}$$

Set

$$\tau_n(t) = e^{-\eta_n/K_1(T-t)} = e^{-(2n+1)/K_1(T-t)} \tag{C.12}$$

In the light of (3.9), for each integer M the function

$$w_M(t,z) \equiv \sum_{n=0}^{M} \alpha_n\tau_n(t)\psi_n(z) \tag{C.13}$$

solves the differential expression in (C.11) with boundary condition $w_M(T,z) = \sum_{n=0}^{M}\alpha_n\psi_n(z)$. Thus, it suffices to prove that the function $w(t,z) = \lim_{M\uparrow\infty} w_M(t,z)$ is a well defined $C^{1,2}((0,T)\times\mathbb{R})$ function, which is strictly positive and which solves the PDE in (C.11).

Since φ is a function of rapid decrease, taking $l = 0, m = 0$ in (3.12) shows that $w_M(T,z)$ converges uniformly to $\varphi(z)$ on $\mathbb{R}$. To show that $w(t,z) = \lim_{M\uparrow\infty} w_M(t,z)$ exists on $(0,T)\times\mathbb{R}$ and solves the full PDE in (C.11), it is enough to prove any integer l that $\frac{d^l}{dz^l}w_M(t,z)$ as well as $\partial_t w_M(t,z)$ are uniformly convergent as $M \uparrow \infty$ in $(0,T)\times\mathbb{R}$. Regarding the spatial derivatives for $l = 1$ it is clear that for each $t \le T$, $\gamma_n = \tau_n(t)$ satisfies the hypothesis of Lemma C.2 with

$\tau_n(t) \le e^{-T/K_1}$ for all n, $0 \le t \le T$. Since $w_M(T,z)$ converges uniformly in $\mathbb{R}$ for any $\varepsilon > 0$ there is an M_ε such that $m, M > M_\varepsilon$ implies that

$$\sup_{z\in\mathbb{R}} \left| \sum_{n=m}^{M} \alpha_n \psi_n(z) \right| < \varepsilon \tag{C.14}$$

Hence, by Lemma C.2 for any $0 \le t \le T$

$$\sup_{z\in\mathbb{R}} \left| \sum_{n=m}^{M} \alpha_n \tau_n(t) \psi_n(z) \right| \le e^{-T/K_1}\varepsilon$$

and thus $w_M(t,y)$ converges uniformly on $(0,T)\times\mathbb{R}$. The proof for $\frac{d^l}{dz^l} w_M(t,z)$ for any l is analogous, using the convergence in the space of functions of rapid decrease at $t=T$. As for $\partial_t w_M$ note that:

$$\partial_t w_M = \sum_{n=0}^{M} \frac{\eta_n}{K_1} \tau_n(t) \alpha_n \psi_n(z) = \frac{1}{K_1} \sum_{n=0}^{M} \tau_n(t) \alpha_n \left(-\ddot{\psi}_n(z) + z^2 \psi_n(z) \right)$$

since $\eta_n \psi_n = -\ddot{\psi}_n + z^2 \psi_n$. Thus, the uniform convergence of $\partial_t w_M$ follows from that of $\dot{w}_M$ and $z^2 \ddot{w}_M$. This proves that w satisfies the PDE in (C.11). Note that (C.11) can be written

$$\begin{aligned} w_t + L w_{zz} - \frac{z^2}{K_1} w = 0 \qquad & (t,z) \in (0,T)\times\mathbb{R} \\ w(T,z) = \varphi(z) \qquad & z \in \mathbb{R} \end{aligned} \tag{C.15}$$

for the operator $L = \frac{1}{2}\frac{2}{K_1} d^2/dz^2$. Thus, using the fact that w solves (C.11) and that φ is a function of rapid decrease it clearly follows that $w(t,z)$ admits the stochastic representation

$$w(t,z) = E_z^{\mathbb{Q}} \left[\varphi(Z_{T-t}) \exp\left(-\frac{1}{K_1} \int_0^{T-t} Z_s^2 ds \right) \right] \tag{C.16}$$

where $(\mathbb{Q}_z)_{z\in\mathbb{R}}$ is a solution to the Martingale Problem for L on $\mathbb{R}$. Such a solution clearly exists since $Z = \sqrt{2/K_1} W$ where W is a standard Brownian Motion. The strict positivity of $w(t,z)$ easily follows from (C.16) since $\varphi > 0$.

It is next show that v satisfies the convergence relation in (3.14). By (C.10) it follows that at $z = \alpha y + \beta$

$$\frac{v_y(t,y)}{v(t,y)} = \alpha \left(\frac{w_z(t,z)}{w(t,z)} - \frac{\dot{\varphi}(z)}{\varphi(z)} \right) \tag{C.17}$$

By (3.6) and (3.10) it follows that at $z = \alpha y + \beta$, $\phi_0(y) = \varphi(z)^{-1} \psi_0(z)$ and hence

$$\frac{v_y(t,y)}{v(t,y)} - \frac{\dot{\phi}_0(y)}{\phi_0(y)} = \zeta \left(\frac{w_z(t,z)}{w(t,z)} - \frac{\dot{\psi}_0(z)}{\psi_0(z)} \right)$$

and thus (3.14) will follow if

$$\lim_{T\uparrow\infty} \frac{w_z(t,z)}{w(t,z)} = \frac{\dot{\psi}_0(z)}{\psi_0(z)} \tag{C.18}$$

holds for all $(t,z) \in (0,\infty)\times\mathbb{R}$. Using the uniform convergence of $w_M, \dot{w}_M$ it follows that

$$\begin{aligned} \lim_{T\uparrow\infty} \frac{w_z(t,z)}{w(t,z)} &= \lim_{T\uparrow\infty} \frac{\sum_{n=0}^{\infty} \tau_n(t) \alpha_n \dot{\psi}_n(z)}{\sum_{n=0}^{\infty} \tau_n(t) \alpha_n \psi_n(z)} \\ &= \lim_{T\uparrow\infty} \frac{\alpha_0 \dot{\psi}_0(z) + \sum_{n=1}^{\infty} (\tau_n(t)/\tau_0(t)) \alpha_n \dot{\psi}_n(z)}{\alpha_0 \psi_0(z) + \sum_{n=1}^{\infty} (\tau_n(t)/\tau_0(t)) \alpha_n \psi_n(z)} \end{aligned}$$

Since $\tau_n(t)/\tau_0(t) = e^{-2n/K_1(T-t)}$ it suffices to show

$$\lim_{T\uparrow\infty}\lim_{M\uparrow\infty}\sum_{n=1}^{M} e^{-2n/K_1(T-t)}\alpha_n\dot{\psi}_n(z) = 0 \qquad \lim_{T\uparrow\infty}\lim_{M\uparrow\infty}\sum_{n=1}^{\infty} e^{-2n/K_1(T-t)}\alpha_n\psi_n(z) = 0$$

It will be shown for the partial sums of ψ_n, the proof for the partial sums of $\dot{\psi}_n$ being similar. Fix $t > 0$ and $z \in \mathbb{R}$. Let $\tilde{T}$ be such that $t < \tilde{T} < T$. Since $w_M(t, z)$ converges uniformly in z for $\tilde{T}$ for any $\varepsilon > 0$ there is an M_ε such that for $m, M > M_\varepsilon$

$$\left|\sum_{n=m}^{M} e^{-\eta_n/K_1(\tilde{T}-t)}\alpha_n\psi_n(z)\right| < \varepsilon$$

By making ε smaller it can be assumed that, plugging in $\eta_n = 2n + 1$,

$$\left|\sum_{n=m}^{M} e^{-2n/K_1(\tilde{T}-t)}\alpha_n\psi_n(z)\right| < \varepsilon$$

It thus follows using Lemma C.2 with $\gamma_n = e^{-2n/K_1(T-\tilde{T})}$ and $h_n = e^{-2n/K_1(\tilde{T}-t)}\alpha_n\psi_n(z)$ that

$$\begin{aligned}
\lim_{T\uparrow\infty}\lim_{M\uparrow\infty}\left|\sum_{n=1}^{M} e^{-2n/K_1(T-t)}\alpha_n\psi_n(z)\right| &\le \lim_{T\uparrow\infty}\lim_{M\uparrow\infty}\left|\sum_{n=M_\varepsilon+1}^{M} e^{-2n/K_1(T-t)}\alpha_n\psi_n(z)\right| \\
&= \lim_{T\uparrow\infty}\lim_{M\uparrow\infty}\left|\sum_{n=M_\varepsilon+1}^{M} e^{-2n/K_1(T-\tilde{T})}e^{-2n/K_1(\tilde{T}-t)}\alpha_n\psi_n(z)\right| \\
&\le \lim_{T\uparrow\infty}\lim_{M\uparrow\infty} e^{-2(M_\varepsilon+1)/K_1(T-\tilde{T})} \max_{N=M_\varepsilon+1,\dots,M}\left|\sum_{n=N}^{M} e^{-2n/K_1(\tilde{T}-t)}\alpha_n\psi_n(z)\right| \\
&\le \lim_{T\uparrow\infty} \varepsilon e^{-2(M_\varepsilon+1)/K_1(T-\tilde{T})} \\
&= 0
\end{aligned}$$

It is now shown using Lemma B.3 that the value function $V(x, t, y)$ is equal to $\frac{x^p}{p}v(t, y)^\delta$. It has already been shown that v satisfies the PDE in (B.1) specified to (3.1). Thus, it only remains to prove that the model $\hat{P}_T$ satisfies Assumption B.2. Specified to the model in (3.1), $\hat{P}_T$ takes the form.

$$\begin{aligned}
dR_t &= \frac{1}{1-p}\left(\sigma\nu_0 + b\sigma\nu_1 Y_t + \delta\sigma\rho\frac{v_y}{v}(t, Y_t)\right)dt + \sigma dZ_t \\
dY_t &= \left(-q\rho'\nu_0 - b\left(1 + q\rho'\nu_1\right)Y_t + \frac{v_y}{v}(t, Y_t)\right)dt + dW_t
\end{aligned}$$

Notice that it is enough to prove there is a weak solution for the SDE involving Y. Indeed, if this is the case, there is a weak solution for Y, B where B is an n-dimensional Brownian Motion independent of Y. Then, setting $Z = \rho W + \bar{\rho}B$ and defining R accordingly will result in a weak solution for R, Y.

Clearly, for each $z \in \mathbb{R}$, there is a weak solution to the SDE

$$dZ_t = \sqrt{\frac{2}{K_1}}dW_t \qquad Z_0 = z$$

Indeed, the measures $(\mathbb{Q}_z)_{z\in\mathbb{R}}$ from (C.16) provide such a solution. Using (C.16) it follows that

$$\frac{1}{w(0,z)}E_z^{\mathbb{Q}}\left[\varphi(Z_T)\exp\left(-\frac{1}{K_1}\int_0^T Z_t^2 dt\right)\right] = 1$$

But, Itô's formula implies that under $\mathbb{Q}_z$

$$\frac{1}{w(0,z)}\varphi(Z_T)\exp\left(-\frac{1}{K_1}\int_0^T Z_t^2 dt\right) = \mathcal{E}\left(\int_0^{\cdot}\sqrt{\frac{2}{K_1}}\frac{w_z}{w}(t,Z_t)dW_t\right)_T$$

and hence by Girsanov's theorem there is a weak solution to the SDE

$$\begin{aligned} dZ_t &= \frac{2}{K_1}\frac{w_z}{w}(t,Z_t)dt + \sqrt{\frac{2}{K_1}}dW_t \\ &= \alpha^2\frac{w_z}{w}(t,Z_t)dt + \alpha dW_t \end{aligned}$$

using the definition of α. Setting $Y = \frac{1}{\alpha}(Z-\beta)$ and using (C.17), (3.10) and (3.3) gives

$$\begin{aligned} dY_t &= \alpha\frac{w_z}{w}(t,\alpha Y_t+\beta)dt + dW_t \\ &= \left(\frac{v_y}{v}(t,Y_t) + \alpha\frac{\dot{\varphi}}{\varphi}(Y_t)\right)dt + dW_t \\ &= \left(-q\rho'\nu_0 - b(1+q\rho'\nu_1)Y_t + \frac{v_y}{v}(t,Y_t)\right)dt + dW_t \end{aligned}$$

Therefore the model $\hat{P}_T$ is well posed.

Lastly, the statements regarding (3.15) follow immediately from the fact that $\psi_0(z) = \pi^{-1/4}e^{-z^2/2}$ and that $\phi_0(y) = \sqrt{\alpha}m(y)^{-1/2}\psi_0(\alpha y+\beta)$.

It remains to show the static fund separation given in (3.15). By (3.6) and the fact that $\psi_0(z) = \pi^{-1/4}e^{-z^2/2}$ we have

$$\frac{\dot{\phi}_0(y)}{\phi_0(y)} = \left(q\rho'\nu_0 - \alpha\beta\right)\times 1 + \left(b(1+q\rho'\nu_1) - \alpha^2\beta\right)\times y$$

from which (3.15) follows by plugging in for the constants in (C.1). □

D Proofs of Section 4

The following constants are used in the proof of Lemma 4.4 and Theorem 4.5.

$$\begin{aligned} &\alpha = b\theta - qa\rho'\nu_0 - \frac{1}{2}a^2; \qquad \beta = b + qa\rho'\nu_1 \\ &\Lambda = \alpha^2 + \frac{qa^2}{\delta}\nu_0'\nu_0; \qquad \Theta = \beta^2 - \frac{2a^2}{\delta}\left(pr_1 - \frac{1}{2}q\nu_1'\nu_1\right) \\ &\omega = \frac{2\sqrt{\Lambda}}{a^2}; \qquad \zeta = \frac{2\sqrt{\Theta}}{a^2}; \qquad K_1 = \frac{1}{\sqrt{\Theta}} \\ &K_2 = \frac{a^2}{2\sqrt{\Theta}}\left(\frac{\beta-\sqrt{\Theta}}{a^2} + \frac{2\left(\alpha\beta - \sqrt{\Lambda\Theta}\right)}{a^4} + \frac{2}{a^2\delta}\left(pr_0 - q\nu_0'\nu_1\right)\right) \end{aligned} \tag{D.1}$$

The proof of Lemma 4.4 and Theorem 4.5 requires the following three lemmas.

Lemma D.1. *Let Assumption 4.1 hold. Let m be as in* (4.3) *and $\hat{m}$ be as in* (4.7). *Let ω, ζ, K_1 and K_2 be as in* (D.1). *Let $\lambda \in \mathbb{R}$ and $\eta = K_1\lambda + K_2$. Let*

$$V(y) = \frac{1}{\delta}\left(pr_0 + pr_1 y - \frac{1}{2}q\nu_0'\nu_0\frac{1}{y} - q\nu_0'\nu_1 - \frac{1}{2}q\nu_1'\nu_1 y\right) \tag{D.2}$$

If $f, g \in C^{1,2}((0,T)\times(0,\infty))$ are related by

$$f(t,y)m(y)^{1/2} = \sqrt{\zeta}g(t,z)\hat{m}(z)^{1/2} \iff f(t,y) = g(t,z)\varphi(z)^{-1}$$

for $z = \zeta y$ then at $z = \zeta y$

$$\begin{aligned}\frac{1}{2}&\left(a^2 y\dot{f}(t,y)m(y)\right)^{\cdot} + V(y)f(t,y)m(y) + \lambda f(t,y)m(y) + f_t(t,y)m(y)\\ &= \frac{1}{2}a^2\zeta^{3/2}\sqrt{\hat{m}(z)m(y)}\left(z\ddot{g}(t,z) + (1+\omega-z)\dot{g}(t,z) + \eta g(t,z) + \frac{1}{\sqrt{\Theta}}g_t(t,z)\right)\end{aligned} \tag{D.3}$$

Proof of Lemma D.1. That $f_t(t,y)m(y) = \frac{1}{2}a^2\zeta^{3/2}(1/\sqrt{\Theta})\sqrt{m(y)\hat{m}(z)}g_t(t,z)$ is immediate since $\sqrt{\Theta} = (a^2/2)\zeta$. For the remaining terms it suffices to think of f, g as functions of y (resp z) alone. Define $\Xi(y)$ by

$$\Xi(y) = \frac{1}{2}\left(a^2 y\dot{f}(y)m(y)\right)^{\cdot} + V(y)f(y)m(y) + \lambda f(y)m(y) \tag{D.4}$$

From (D.1) and (4.3) it follows that $\frac{\dot{m}}{2m} = \frac{\alpha}{a^2 y} - \frac{\beta}{a^2}$. Plugging this in gives (note: we suppress the y from the right hand side of the following equations to ease notation)

$$\begin{aligned}\Xi(y) &= m\left(\frac{1}{2}a^2\dot{f} + \frac{1}{2}a^2 y\ddot{f} + a^2 y\dot{f}\frac{\dot{m}}{2m} + (V+\lambda)f\right)\\ &= m\left(\frac{1}{2}a^2 y\ddot{f} + \left(\alpha + \frac{1}{2}a^2 - \beta y\right)\dot{f} + (V+\lambda)f\right)\end{aligned} \tag{D.5}$$

Let $h(y) = \sqrt{m(y)/\hat{m}(\zeta y)}f(y) = (1/K)y^{-\mathbf{A}}e^{-\mathbf{B}y}f(y)$ where ζ is from (D.1), K is a normalizing constant and

$$\mathbf{A} = 1/a^2\left(\sqrt{\Lambda} - \alpha\right) \qquad \mathbf{B} = 1/a^2\left(-\sqrt{\Theta} + \beta\right) \tag{D.6}$$

It follows that

$$\begin{aligned}\dot{f} &= Ky^{\mathbf{A}}e^{\mathbf{B}y}\left(\frac{\mathbf{A}}{y}h + \mathbf{B}h + \dot{h}\right)\\ \ddot{f} &= Ky^{\mathbf{A}}e^{\mathbf{B}y}\left(\frac{\mathbf{A}(\mathbf{A}-1) + 2\mathbf{A}\mathbf{B}y + \mathbf{B}^2y^2}{y^2}h + \frac{2\mathbf{A} + 2\mathbf{B}y}{y}\dot{h} + \ddot{h}\right)\end{aligned}$$

Plugging this into (D.5), plugging in for V from (D.2), and collecting terms yields

$$\begin{aligned}\Xi(y) &= \frac{1}{2}a^2 mKy^{\mathbf{A}}e^{\mathbf{B}y}\left(y\ddot{h} + \left(\left(2\mathbf{A} + \frac{2\alpha}{a^2} + 1\right) + \left(2\mathbf{B} - \frac{2\beta}{a^2}\right)y\right)\dot{h} + \left(\frac{\hat{\mathbf{A}}}{y} + \hat{\mathbf{B}}y + \hat{\mathbf{C}} + \frac{2\lambda}{a^2}\right)h\right)\\ &= \frac{1}{2}a^2\sqrt{\hat{m}(\zeta y)m(y)}\left(y\ddot{h} + \left(\left(2\mathbf{A} + \frac{2\alpha}{a^2} + 1\right) + \left(2\mathbf{B} - \frac{2\beta}{a^2}\right)y\right)\dot{h} + \left(\frac{\hat{\mathbf{A}}}{y} + \hat{\mathbf{B}}y + \hat{\mathbf{C}} + \frac{2\lambda}{a^2}\right)h\right)\end{aligned} \tag{D.7}$$

for

$$\hat{\mathbf{A}} = \mathbf{A}^2 + \frac{2\alpha}{a^2}\mathbf{A} - \frac{q}{\delta a^2}\nu_0'\nu_0$$
$$\hat{\mathbf{B}} = \mathbf{B}^2 - \frac{2\beta}{a^2}\mathbf{B} + \frac{2}{\delta a^2}\left(pr_1 - \frac{q}{2}\nu_1'\nu_1\right)$$
$$\hat{\mathbf{C}} = \mathbf{B} + 2\mathbf{A}\mathbf{B} + \frac{2\alpha\mathbf{B} - 2\beta\mathbf{A}}{a^2} + \frac{2}{a^2\delta}\left(pr_0 - q\nu_0'\nu_1\right)$$

By (D.6) it follows that $\hat{\mathbf{A}} = \hat{\mathbf{B}} = 0$ and

$$\hat{\mathbf{C}} = \frac{\beta - \sqrt{\Theta}}{a^2} + \frac{2\left(\alpha\beta - \sqrt{\Lambda\Theta}\right)}{a^4} + \frac{2}{a^2\delta}\left(pr_0 - q\nu_0'\nu_1\right)$$

Plugging this into (D.7) and using the expressions for K_1, K_2 given in (D.1) yields

$$\Xi(y) = \frac{1}{2}a^2\sqrt{\hat{m}(\zeta y)m(y)}\left(y\ddot{h} + \left(\left(1 + \frac{2\sqrt{\Lambda}}{a^2}\right) - \zeta y\right)\dot{h} + \zeta\left(K_1\lambda + K_2\right)h\right)$$

Setting $z = \zeta y$, $g(z) = (1/\sqrt{\zeta})h(y)$ and $\eta = K_1\lambda + K_2$ yields at $z = \zeta y$ that

$$\Xi(y) = \frac{1}{2}a^2\zeta^{3/2}\sqrt{\hat{m}(z)m(y)}\left(z\ddot{g} + \left(1 + \frac{2\sqrt{\Lambda}}{a^2} - z\right)\dot{g} + \eta g\right)$$

the desired result. □

Lemma D.2. *Let Assumptions 4.1 and 4.2 hold. Let the functions* $(\psi_n)_{n\geq 0}$ *be from* (4.10). *Let* $(\gamma_n)_{n\geq 0}$ *be a sequence of real numbers such that*

$$\sum_{n=0}^{\infty} n^s|\gamma_n| < \infty \tag{D.8}$$

for some $s > \frac{\omega}{2} + \frac{1}{4}$. *Then*

$$\sum_{n=0}^{\infty} \gamma_n\psi_n(z) \tag{D.9}$$

converges uniformly on compact subsets of $(0, \infty)$.

Proof of Lemma D.2. By Hille (1926) it holds that the series $\sum_{n=1}^{\infty} n^{-s}\psi_n(z)$ converges uniformly on compact subsets of $(0, \infty)$. For any positive integers $m < M$

$$\left|\sum_{n=m}^{M} \gamma_n\psi_n(z)\right| = \left|\sum_{n=m}^{M} n^s\gamma_n n^{-s}\psi(z)\right| \leq \max_{n=m,\ldots M}\left|n^{-s}\psi_n(z)\right| \sum_{n=m}^{M} n^s\left|\gamma_n\right|$$

and thus by (D.8) the result follows.

□

Lemma D.3. *Let* $\varphi \in L^2((0,\infty), \hat{m})$ *be strictly positive and continuously differentiable. Let the functions* $(\psi_n)_{n\geq 0}$ *be from* (4.10). *Let*

$$\beta_n = \int_0^{\infty} \varphi(y)\psi_n(y)\hat{m}(y)dy$$

Let $(\mathbb{Q}_z)_{z>0}$ *denote the solution to the martingale problem on* $(0,\infty)$ *for*

$$L = \sqrt{\Theta} z \frac{d^2}{dz^2} + \sqrt{\Theta}\left(1+\omega-z\right)\frac{d}{dz} \tag{D.10}$$

Let Z *denote the coordinate mapping process. Let* $\tau(t) = e^{-\sqrt{\Theta}(T-t)}$*. Then, for* $t \in [0,T]$ *and* $z > 0$

$$\sum_{n=0}^{\infty} \tau(t)^n \beta_n \psi_n(z) = E_z^{\mathbb{Q}}\left[\varphi(Z_{T-t})\right] \tag{D.11}$$

Remark D.4. Note that under $\mathbb{Q}_z$, Z is a CIR process starting at z. Since $\sqrt{\Theta}(1+\omega) > \sqrt{\Theta}$, Z does not hit 0 and thus there is a solution to the martingale problem for L on $(0,\infty)$.

Proof of Lemma D.3. Let $t < T$ and $\gamma_n = \tau(t)^n \beta_n$. Note that for any $s > 0$ by Parseval's equality and the fact that $\sqrt{\Theta} > 0$

$$\sum_{n=0}^{\infty} n^s |\gamma_n| \leq 2\sum_{n=0}^{\infty} n^{2s}\tau(t)^{2n} + 2\sum_{n=0}^{\infty} \beta_n^2 < \infty \tag{D.12}$$

Thus, Lemma D.2 implies the series

$$\sum_{n=0}^{\infty} \tau(t)^n \beta_n \psi_n(z)$$

converges uniformly on compact subsets of $(0,\infty)$. At $t = T$ the pointwise convergence of

$$\sum_{n=0}^{\infty} \beta_n \psi_n(z)$$

follows from Lebedev (1972, Ch 5, Theorem 3) since φ is continuously differentiable. Therefore, the series $\sum_{n=0}^{\infty} \tau(t)^n \beta_n \psi_n(z)$ defines a function on $(0,T) \times (0,\infty)$.

Three facts are used regarding CIR processes and the generalized Laguerre polynomials are used to prove (D.11). First, it is well known (Glasserman, 2004, Ch 3.4) that under $\mathbb{Q}_z$, for $0 \leq t < T$, Z_{T-t} is distributed like KX where

$$K = \frac{1}{2}(1 - e^{-\sqrt{\Theta}(T-t)}) = \frac{1}{2}\left(1 - \tau(t)\right) \tag{D.13}$$

and X is a non-central Chi-Square random variable with non-centrality parameter λ and degrees of freedom d given by

$$\lambda = \frac{2e^{-\sqrt{\Theta}(T-t)} z}{1 - e^{-\sqrt{\Theta}(T-t)}} = \frac{2\tau(t) z}{1-\tau(t)}; \qquad d = 2(1+\omega) \tag{D.14}$$

Secondly, the probability density function for X is given by

$$\begin{aligned} f(x) &= \frac{1}{2} e^{-x/2-\lambda/2} x^{d/4-1/2} \lambda^{-(d/4-1/2)} I_{d/2-1}(\sqrt{\lambda x}) \\ &= \frac{1}{2} e^{-x/2-\lambda/2} x^{\omega/2} \lambda^{-\omega/2} I_{\omega}(\sqrt{\lambda x}) \end{aligned} \tag{D.15}$$

where $I_\omega(y)$ is the modified Bessel function of the first kind. Third, note the following identity regarding generalized Laguerre polynomials: namely, that for $|\tau| < 1$ (Magnus et al., 1966, Chapter 5.5) (note there is a typo there in that it should be $yz\tau$ and not yz/τ)

$$\sum_{n=0}^{\infty} \psi_n(y)\psi_n(z)\tau^n = \frac{\Gamma(1+\omega)}{1-\tau}(yz\tau)^{-\omega/2} \exp\left(-\frac{\tau(y+z)}{1-\tau}\right) I_\omega\left(\sqrt{\frac{4yz\tau}{(1-\tau)^2}}\right) \tag{D.16}$$

(D.15) readily gives that

$$\begin{aligned} E_z^{\mathbb{Q}}\left[\varphi(Z_{T-t})\right] &= \int_0^\infty \varphi(Kx)f(x)dx = \int_0^\infty \varphi(y)f(y/K)\frac{1}{K}dy \\ &= \int_0^\infty \varphi(y)\frac{1}{2K}e^{-y/(2K)-\lambda/2}y^{\omega/2}(\lambda K)^{-\omega/2} I_\omega\left(\sqrt{\frac{\lambda y}{K}}\right) dy \end{aligned} \tag{D.17}$$

Assuming the infinite sum can be interchanged with the integral, which will be shown later, using the definition of $\tau(t)$ and β_n it follows from (D.16) that

$$\begin{aligned} \sum_{n=0}^{\infty} \tau(t)^n \beta_n \psi_n(z) &= \sum_{n=0}^{\infty} \tau(t)^n \psi_n(z) \int_0^\infty \varphi(y)\psi_n(y)\hat{m}(y)dy \\ &= \int_0^\infty \varphi(y)\hat{m}(y) \sum_{n=0}^{\infty} \psi_n(y)\psi_n(z)\tau(t)^n \\ &= \int_0^\infty \varphi(y)\hat{m}(y)\frac{\Gamma(1+\omega)}{1-\tau(t)}(yz\tau(t))^{-\omega/2} \exp\left(-\frac{\tau(t)(y+z)}{1-\tau(t)}\right) \\ &\qquad \times I_\omega\left(\sqrt{\frac{4yz\tau(t)}{(1-\tau(t))^2}}\right) dy \end{aligned} \tag{D.18}$$

Therefore, the result will hold if the integrands in (D.17) and (D.18) are the same. Thus, it must be shown that

$$\begin{aligned} &\varphi(y)\frac{1}{2K}e^{-y/(2K)-\lambda/2}y^{\omega/2}(\lambda K)^{-\omega/2} I_\omega\left(\sqrt{\frac{\lambda y}{K}}\right) \\ &\quad = \varphi(y)\hat{m}(y)\frac{\Gamma(1+\omega)}{1-\tau(t)}(yz\tau(t))^{-\omega/2} \exp\left(-\frac{\tau(t)(y+z)}{1-\tau(t)}\right) I_\omega\left(\sqrt{\frac{4yz\tau(t)}{(1-\tau(t))^2}}\right) \\ &\quad = \frac{\varphi(y)}{1-\tau(t)}e^{-y/(1-\tau(t))}y^{\omega/2}(z\tau(t))^{-\omega/2} \exp\left(-\frac{\tau(t)z}{1-\tau(t)}\right) I_\omega\left(\sqrt{\frac{4yz\tau(t)}{(1-\tau(t))^2}}\right) \end{aligned} \tag{D.19}$$

where the last line comes from plugging in for $\hat{m} = \frac{1}{\Gamma(1+\omega)}y^\omega e^{-y}$ and collecting terms. Using (D.14) and (D.13) the following hold

$$\frac{\lambda y}{K} = \frac{4yz\tau(t)}{(1-\tau(t))^2}; \quad 2K = 1-\tau(t); \quad \frac{\lambda}{2} = \frac{\tau(t)z}{1-\tau(t)}; \quad \lambda K = \tau(t)z$$

Plugging this into the right hand side of (D.19) gives

$$\varphi(y)\frac{1}{2K}e^{-y/(2K)}y^{\omega/2}(\lambda K)^{-\omega/2} \exp\left(-\lambda/2\right) I_\omega\left(\sqrt{\frac{\lambda y}{K}}\right)$$

the desired result. The last step to prove is that the integral and the infinite sum may be interchanged in (D.18). To see this, for each integer M and a fixed $z > 0$ set

$$f_M(y) = \frac{1}{\varphi(y)} \sum_{n=0}^{M} \tau(t)^n \psi_n(z)\psi_n(y)$$

It then follows that switching the integral and infinite sum amounts to showing that

$$\lim_{M\uparrow\infty} \int_0^\infty f_M(y)\varphi^2(y)\hat{m}(y)dy = \int_0^\infty \lim_{M\uparrow\infty} f_M(y)\varphi^2(y)\hat{m}(y)dy$$

Since $\varphi \in L^2((0,\infty),\hat{m})$ it suffices to show that

$$\sup_{M\in\mathbb{N}} \int_0^\infty f_M^2(y)\varphi^2(y)\hat{m}(y)dy < \infty$$

Indeed, if this is the case, then $(f_M)_{M\in\mathbb{N}}$ is uniformly integrable for the finite measure $\varphi^2\hat{m}$. To this end, using the ortho-normality of (ψ_n) in $L^2((0,\infty),\hat{m})$ it follows that

$$\int_0^\infty f_M^2(y)\varphi^2(y)\hat{m}(y)dy = \sum_{n=0}^{M} \tau^{2n}\psi_n^2(z) \leq \sum_{n=0}^{\infty} \tau^{2n}\psi_n^2(z) < \infty$$

since (D.16) gives an explicit formula for $\sum_{n=0}^\infty \tau(t)^{2n}\psi_n(z)^2$.

□

Proof of Lemma 4.4. By applying Lemma D.1 for

$$f(y) = \phi(y); \quad g(z) = \psi(z)$$

it follows that ϕ satisfies (4.4) for some $\lambda \in \mathbb{R}$ if and only if ψ satisfies (4.8) for $\eta = K_1\lambda + K_2$. As for the respective L^2 norms, since

$$\phi(y)m(y)^{1/2} = \sqrt{\zeta}\psi(\zeta y)\hat{m}(\zeta y)^{1/2}$$

with m from (4.3) (normalized to be a probability measure) and $\hat{m}$ from (4.7) it follows that

$$\int_0^\infty \phi(y)^2 m(y)dy = \int_0^\infty \zeta\psi(\zeta y)^2\hat{m}(\zeta y)dy = \int_0^\infty \psi(z)^2\hat{m}(z)dz$$

Thus, $\phi \in L^2((0,\infty),m)$ if and only if $\psi \in L^2((0,\infty),\hat{m})$ and they have the same norm. □

Proof of Theorem 4.5. It is first shown that v from (4.13) is a strictly positive $C^{1,2}((0,T)\times(0,\infty))$ solution to the PDE in (2.18), or equivalently, to the PDE in (2.20), specified to the model in (4.1). We have, using (4.6), (4.7), (4.10) and (4.11) that for any $y > 0$, at $z = \zeta y$

$$v(t,y) = \sum_{n=0}^{\infty} e^{-\lambda_n(T-t)}\alpha_n\phi_n(y) = e^{K_2/K_1(T-t)}\varphi(z)^{-1}\sum_{n=0}^{\infty} e^{-n\sqrt{\Theta}(T-t)}\alpha_n\psi_n(z) \tag{D.20}$$

Set

$$w(t,z) = \sum_{n=0}^{\infty} e^{-n\sqrt{\Theta}(T-t)}\alpha_n\psi_n(z) \tag{D.21}$$

so that (D.20) becomes

$$v(t,y) = e^{K_2/K_1(T-t)}\varphi(z)^{-1}w(t,z) \tag{D.22}$$

By applying Lemma D.1 for

$$\lambda = 0; \quad f(t,y) = v(t,y); \quad g(t,z) = w(t,z)e^{K_2/K_1(T-t)}$$

it follows that $v(t,y)$ solves (2.20) if and only if $w(t,z)$ solves (note $K_1 = 1/\sqrt{\Theta}$)

$$\begin{aligned} w_t + \sqrt{\Theta}zw_{zz} + \sqrt{\Theta}(1+\omega-z)w_z = 0 \qquad & (t,z) \in (0,T)\times(0,\infty) \\ w(T,z) = \varphi(z) \qquad & z \in (0,\infty) \end{aligned} \tag{D.23}$$

Note that (D.23) can be re-written

$$\begin{aligned} w_t + Lw = 0 \qquad & (t,z) \in (0,T)\times(0,\infty) \\ w(T,z) = \varphi(z) \qquad & z \in (0,\infty) \end{aligned}$$

for the operator L from (D.10). Since $\varphi \in L^2((0,\infty),\hat{m})$, Lemma D.3 implies that

$$w(t,z) = \sum_{n=0}^{\infty} e^{-n\sqrt{\Theta}(T-t)}\alpha_n\psi_n(z) = E_z^{\mathbb{Q}}\left[\varphi(Z_{T-t})\right] \tag{D.24}$$

To invoke Proposition A.2 it must be shown that for any $T > 0$

$$\sup_{0\le t\le T} E_z^{\mathbb{Q}}\left[\varphi(Z_{T-t})\right] < \infty$$

To this end, it can be shown using (4.3), (4.7), (4.11) and (D.1) that under Assumptions 4.1 and 4.2 for the operator L in (D.10) that there is a constant K such that

$$L\varphi(z) \le K\varphi(z)$$

Therefore

$$\partial_t\left(e^{K(T-t)}\varphi(z)\right) + L\left(e^{K(T-t)}\varphi(z)\right) \le 0$$

and hence using Itô's formula and Fatou's lemma it follows that

$$\sup_{0\le t\le T} E_z^{\mathbb{Q}}\left[\varphi(Z_{T-t})\right] \le e^{KT}\varphi(z) < \infty$$

Thus, by Proposition A.2 it follows that w satisfies (D.23). The strict positivity of w, and hence v follows from that of φ.

It is next show that v satisfies the convergence relation in (4.14). By (D.22) it follows that at $z = \zeta y$

$$\frac{v_y(t,y)}{v(t,y)} = \zeta\left(\frac{w_z(t,z)}{w(t,z)} - \frac{\dot{\varphi}(z)}{\varphi(z)}\right) \tag{D.25}$$

Now, by (4.9) it follows that $\psi_0(z) = 1$. By (4.6) and (4.11) it follows that at $z = \zeta y$, $\phi_0(y) = \varphi(z)^{-1}$ and hence

$$\frac{v_y(t,y)}{v(t,y)} - \frac{\dot{\phi}_0(y)}{\phi_0(y)} = \zeta\frac{w_z(t,z)}{w(t,z)} = \zeta\left(\frac{w_z(t,z)}{w(t,z)} - \frac{\dot{\psi}_0(z)}{\psi_0(z)}\right)$$

and thus (4.14) will follow if

$$\lim_{T\uparrow\infty}\frac{w_z(t,z)}{w(t,z)}=\frac{\dot{\psi}_0(z)}{\psi_0(z)} \tag{D.26}$$

holds for all $(t,z)\in(0,\infty)\times(0,\infty)$. Set w_M by

$$w_M(t,z)=\sum_{n=0}^{M}\tau(t)^n\alpha_n\psi_n(z)$$

(D.26) will follow if w_M and $\dot{w}_M$ locally (in z) converge uniformly as $M\uparrow\infty$. By repeating the argument at the beginning of the proof of Lemma D.3 it follows that for $t<T$, $w_M(t,z)$ converges uniformly in z on compact subsets of $(0,\infty)$. As for $\dot{w}_M(t,z)$, the following recurrence relation for the Laguerre polynomials is needed (Hille, 1926)

$$z\dot{\psi}_n=(n+1)\psi_{n+1}-(n+1+\omega-z)\psi_n \tag{D.27}$$

This gives for each $z>0$

$$\begin{aligned}\dot{w}_M(t,z)&=\frac{1}{z}\sum_{n=0}^{M}\tau(t)^n\alpha_n\left((n+1)\psi_{n+1}(z)-(n+1+\omega-z)\psi_n(z)\right)\\&=\frac{1}{z}\sum_{n=0}^{M}\alpha_n\tau(t)^n(n+1)\psi_{n+1}(z)-\frac{1}{z}\sum_{n=0}^{M}\tau(t)^n n\alpha_n\psi_n(z)-\frac{1}{z}(1+\omega-z)w_m(t,z)\end{aligned}$$

It has already been shown that w_M converges uniformly in z on compact subsets of $(0,\infty)$. As for the other two sums

$$\sum_{n=0}^{M}\alpha_n\tau(t)^n(n+1)\psi_{n+1}(z)=\sum_{n=0}^{\infty}\gamma_n\psi_n(z)\qquad\gamma_0=0,\gamma_n=n\alpha_{n-1}\tau(t)^{n-1},n=1,2,\dots$$

$$\sum_{n=0}^{M}\tau(t)^n n\alpha_n\psi_n(z)=\sum_{n=0}^{\infty}\gamma_n\psi_n(z)\qquad\gamma_n=n\alpha_n\tau(t)^n,n=0,1,\dots$$

Thus, Parseval's equality and $\sqrt{\Theta}>0$ clearly imply that (D.8) holds for any $s>0$ uniformly for $t\in(0,T-\varepsilon)$ for any $\varepsilon>0$. Thus, $\dot{w}_m$ converges uniformly for z on compact subsets of $(0,\infty)$ and for $t<T-\varepsilon$ for any $\varepsilon>0$. Using the local uniform convergence of w_M and $\dot{w}_M$ it follows that

$$\begin{aligned}\lim_{T\uparrow\infty}\frac{w_z(t,z)}{w(t,z)}&=\lim_{T\uparrow\infty}\frac{\sum_{n=0}^{\infty}\tau(t)^n\alpha_n\dot{\psi}_n(z)}{\sum_{n=0}^{\infty}\tau(t)^n\alpha_n\psi_n(z)}\\&=\lim_{T\uparrow\infty}\frac{\alpha_0\dot{\psi}_0(z)+\sum_{n=1}^{\infty}\tau(t)^n\alpha_n\dot{\psi}_n(z)}{\alpha_0\psi_0(z)+\sum_{n=1}^{\infty}\tau(t)^n\alpha_n\psi_n(z)}\end{aligned}$$

and thus it suffices to show

$$\lim_{T\uparrow\infty}\lim_{M\uparrow\infty}\sum_{n=1}^{M}\tau(t)^n\alpha_n\dot{\psi}_n(z)=0\qquad\lim_{T\uparrow\infty}\lim_{M\uparrow\infty}\sum_{n=1}^{\infty}\tau(t)^n\alpha_n\psi_n(z)=0$$

The limit will be shown for the first sum. The proof for the second is the exact same. Fix $t>0$, $z>0$ and let $\tilde{T}$ be such that $t<\tilde{T}<T$. By the local uniform convergence of $\dot{w}_M$ for $\tilde{T}$ instead of T, for any $\epsilon>0$ there is a M_ε such that $m,M>M_\varepsilon$ imply that

$$\max_{N=m,\dots,M}\left|\sum_{n=m}^{N}e^{-n\sqrt{\Theta}(\tilde{T}-t)}\alpha_n\dot{\psi}_n(z)\right|<\varepsilon$$

It thus follows using Lemma C.2 with $\gamma_n = e^{-n\sqrt{\theta}(T-\tilde{T})}$ and $h_n = e^{-n\sqrt{\Theta}(\tilde{T}-t)}\alpha_n\dot{\psi}_n(z)$ that

$$\begin{aligned}
\lim_{T\uparrow\infty}\lim_{M\uparrow\infty}\left|\sum_{n=1}^{M}\tau(t)^n\alpha_n\dot{\psi}_n(z)\right| &\leq \lim_{T\uparrow\infty}\lim_{M\uparrow\infty}\left|\sum_{n=M_\varepsilon+1}^{M}\tau(t)^n\alpha_n\dot{\psi}_n(z)\right| \\
&= \lim_{T\uparrow\infty}\lim_{M\uparrow\infty}\left|\sum_{n=M_\varepsilon+1}^{M} e^{-n\sqrt{\Theta}(T-\tilde{T})}e^{-n\sqrt{\Theta}(\tilde{T}-t)}\alpha_n\dot{\psi}_n(z)\right| \\
&\leq \lim_{T\uparrow\infty} e^{-(M_\varepsilon+1)\sqrt{\Theta}(T-\tilde{T})}\max_{N=m,\dots,M}\left|\sum_{n=m}^{N} e^{-n\sqrt{\Theta}(\tilde{T}-t)}\alpha_n\dot{\psi}_n(z)\right| \\
&\leq \varepsilon\lim_{T\uparrow\infty} e^{-(M_\varepsilon+1)\sqrt{\Theta}(T-\tilde{T})} \\
&= 0
\end{aligned}$$

It is now shown using Lemma B.3 that the value function $V(x,t,y)$ is equal to $\frac{x^p}{p}v(t,y)^\delta$. It has already been shown that v satisfies the PDE in (B.1) specified to (4.1). Thus, it only remains to prove that the model $\hat{P}_T$ satisfies Assumption B.2. Specified to the model in (4.1), $\hat{P}_T$ takes the form.

$$\begin{aligned}
dR_t &= \frac{1}{1-p}\left(\sigma\nu_0 + \sigma\nu_1 Y_t + \delta Y_t\sigma\rho a\frac{v_y}{v}(t,Y_t)\right)dt + \sqrt{Y_t}\sigma dZ_t \\
dY_t &= \left(\alpha + \frac{a^2}{2} - \beta Y_t + a^2 Y_t\frac{v_y}{v}(t,Y_t)\right)dt + a\sqrt{Y_t}dW_t
\end{aligned}$$

Notice that it is enough to prove there is a weak solution for the SDE involving Y. Indeed, if this is the case, there is a weak solution for Y, B where B is an n-dimensional Brownian Motion independent of Y. Then, setting $Z = \rho W + \bar{\rho}B$ and defining R accordingly will result in a weak solution for R, Y.

Regarding Y, note first that clearly there is a weak solution to the SDE

$$dZ_t = \sqrt{\Theta}\left(1+\omega-Z_t\right)dt + \sqrt{2\sqrt{\Theta}}\sqrt{Z_t}dW_t$$

Indeed, the operator L associated to this solution is given in (D.10) in Lemma D.3. Let $(\mathbb{Q}_z)_{z>0}$ denote the solution to the martingale problem for L on $(0,\infty)$. From Proposition A.2, equation (A.4), and from (D.24) it follows that $w(t,Z_t)/w(0,z), 0 \leq t \leq T$ is a $\mathbb{Q}_z$ martingale and

$$\frac{w(t,Z_t)}{w(0,z)} = \mathcal{E}\left(\int_0^{\cdot}\sqrt{2\sqrt{\Theta}}\sqrt{Z_s}\frac{w_z}{w}(t,Z_s)dW_t\right)_t$$

Therefore, by Girsanov's theorem, there is a weak solution to the SDE

$$dZ_t = \left(\sqrt{\Theta}(1+\omega) - \sqrt{\Theta}Z_t + 2\sqrt{\Theta}Z_t\frac{w_z}{w}(t,Z_t)\right)dt + \sqrt{2\sqrt{\Theta}}\sqrt{Z_t}dW_t \tag{D.28}$$

Now, using (4.11) and (D.1) it follows that

$$dZ_t = \left(\zeta\left(\alpha + \frac{a^2}{2}\right) - \beta Z_t + a^2\zeta Z_t\left(\frac{w_z}{w}(t,Z_t) - \frac{\dot{\varphi}}{\varphi}(Z_t)\right)\right)dt + a\sqrt{\zeta}\sqrt{Z_t}dW_t$$

Using (D.25) it follows that

$$dZ_t = \left(\zeta\left(\alpha + \frac{a^2}{2}\right) - \beta Z_t + a^2 Z_t\frac{v_y}{v}(t,Z_t/\zeta)\right)dt + a\sqrt{\zeta}\sqrt{Z_t}dW_t$$

Setting $Y = Z/\zeta$ it follows that

$$dY_t = \left(\alpha + \frac{a^2}{2} - \beta Y_t + a^2 Y_t \frac{v_y}{v}(t, Y_t)\right) dt + a\sqrt{Y_t} dW_t$$

Therefore, there is a weak solution for Y and hence (R, Y) with dynamics given by the model $\hat{P}_T$.

It remains to show the static fund separation given in (4.15). By (4.6) and the fact that $\psi_0 = 1$ we have

$$\frac{\dot{\phi}_0(y)}{\phi_0(y)} = -\zeta \frac{\dot{\varphi}(\zeta y)}{\varphi(\zeta y)}$$

Plugging in for $\dot{\varphi}/\varphi$ according to (4.11) and (D.1) gives

$$\frac{\dot{\phi}_0(y)}{\phi_0(y)} = -\zeta \frac{\dot{\varphi}(\zeta y)}{\varphi(\zeta y)} = \frac{\sqrt{\Lambda} - \alpha}{a^2} \times \frac{1}{y} + \frac{\beta - \sqrt{\Theta}}{a^2} \times 1$$

Using (D.1) again yields (4.15).

□

References

Barberis, N. (2000), 'Investing for the Long Run When Returns Are Predictable', *The Journal of Finance* **55**(1), 225–264.

Brandt, M. (1999), 'Estimating Portfolio and Consumption Choice: A Conditional Euler Equations Approach', *The Journal of Finance* **54**(5), 1609–1645.

Cass, D. and Stiglitz, J. E. (1970), 'The structure of investor preferences and asset returns, and separability in portfolio allocation: a contribution to the pure theory of mutual funds', *J. Econom. Theory* **2**, 122–160.

Chamberlain, G. (1988), 'Asset pricing in multiperiod securities markets', *Econometrica* **56**(6), 1283–1300.
URL: *http://dx.doi.org/10.2307/1913098*

Cheridito, P., Filipović, D. and Yor, M. (2005), 'Equivalent and absolutely continuous measure changes for jump-diffusion processes', *Ann. Appl. Probab.* **15**(3), 1713–1732.

Cox, J. C. and Huang, C.-f. (1992), 'A continuous-time portfolio turnpike theorem', *J. Econom. Dynam. Control* **16**(3-4), 491–507.

Cox, J. C., Ingersoll, Jr., J. E. and Ross, S. A. (1985), 'A theory of the term structure of interest rates', *Econometrica* **53**(2), 385–407.
URL: *http://dx.doi.org/10.2307/1911242*

Dybvig, P. H., Rogers, L. and Back, K. (1999), 'Portfolio turnpikes', *Review of Financial Studies* **12**(1).

Feller, W. (1951), 'Two singular diffusion problems', *Ann. of Math. (2)* **54**, 173–182.

Friedman, A. (1964), *Partial differential equations of parabolic type*, Prentice-Hall Inc., Englewood Cliffs, N.J.

Friedman, A. (1975), *Stochastic differential equations and applications. Vol. 1*, Academic Press [Harcourt Brace Jovanovich Publishers], New York. Probability and Mathematical Statistics, Vol. 28.

Glasserman, P. (2004), *Monte Carlo methods in financial engineering*, Vol. 53 of *Applications of Mathematics (New York)*, Springer-Verlag, New York. Stochastic Modelling and Applied Probability.

Guasoni, P. and Robertson, S. (2010), 'Portfolios and Risk Premia for the Long Run', *Working Paper* .

Hakansson, N. H. (1974), 'Convergence to isoelastic utility and policy in multiperiod portfolio choice', *Journal of Financial Economics, 1* **1**, 201–224.

Heath, D. and Schweizer, M. (2000), 'Martingales versus PDEs in finance: an equivalence result with examples', *Journal of Applied Probability* **37**, 947–957.

Heston, S. (1993), 'A closed-form solution for options with stochastic volatility with applications to bond and currency options', *Review of financial studies* **6**(2), 327–343.

Hille, E. (1926), 'On laguerre's series, second note', *Proceedings of the National Academy of Sciences of the United States of America* **12**(4), 265–269.

Huang, C.-f. and Zariphopoulou, T. (1999), 'Turnpike behavior of long-term investments', *Finance Stoch.* **3**(1), 15–34.

Huberman, G. and Ross, S. (1983), 'Portfolio turnpike theorems, risk aversion, and regularly varying utility functions', *Econometrica* **51**(5), 1345–1361.

Khanna, A. and Kulldorff, M. (1999), 'A generalization of the mutual fund theorem', *Finance Stoch.* **3**(2), 167–185.
URL: *http://dx.doi.org/10.1007/s007800050056*

Kim, T. and Omberg, E. (1996), 'Dynamic Nonmyopic Portfolio Behavior', *The Review of Financial Studies* **9**(1), 141–161.

Kramkov, D. and Schachermayer, W. (1999), 'The asymptotic elasticity of utility functions and optimal investment in incomplete markets', *Ann. Appl. Probab.* **9**(3), 904–950.

Lebedev, N. N. (1972), *Special functions and their applications*, Dover Publications Inc., New York. Revised edition, translated from the Russian and edited by Richard A. Silverman, Unabridged and corrected republication.

Leland, H. (1972), On turnpike portfolios, *in* G. Szego and K. Shell, eds, 'Mathematical Models in Investment and Finance', North-Holland, Amsterdam.

Magnus, W., Oberhettinger, F. and Soni, R. P. (1966), *Formulas and theorems for the special functions of mathematical physics*, Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52, Springer-Verlag New York, Inc., New York.

Merton, R. (1973), 'An Intertemporal Capital Asset Pricing Model', *Econometrica* **41**(5), 867–887.

Miklavčič, M. (1998), *Applied functional analysis and partial differential equations*, World Scientific Publishing Co. Inc., River Edge, NJ.

Pinsky, R. G. (1995), *Positive harmonic functions and diffusion*, Vol. 45 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge.

Reed, M. and Simon, B. (1972), *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York.

Ross, S. A. (1978), 'Mutual fund separation in financial theory—the separating distributions', *J. Econom. Theory* **17**(2), 254–286.

Schachermayer, W., Sîrbu, M. and Taflin, E. (2009), 'In which financial markets do mutual fund theorems hold true?', *Finance Stoch.* **13**(1), 49–77.
URL: *http://dx.doi.org/10.1007/s00780-008-0072-x*

Tobin, J. (1958), 'Liquidity preference as behavior towards risk', *The Review of Economic Studies* **25**(2), 65–86.

Wachter, J. (2002), 'Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete markets', *Journal of Financial and Quantitative Analysis* **37**(1), 63–91.

Zariphopoulou, T. (2001), 'A solution approach to valuation with unhedgeable risks', *Finance Stoch.* **5**(1), 61–82.