46-679 Multi-Period Pricing Models
Solutions to Homework Set 4
Due August 10, 1999

Chapter 2, Exercise 11. Consider an N-period binomial model. An *Asian option* has a payoff based on the average stock price, i.e.,

$$V_N = f\left(\frac{1}{N+1}\sum_{n=0}^{N} S_n\right),$$

where the function f is determined by the contractual details of the option.

(i) Define $Y_n = \sum_{k=0}^{n} S_k$ and use the Independence Lemma 5.3 to show that the two-dimensional process (S_n, Y_n), $n = 0, 1, \ldots, N$, is Markov.

(ii) According to Theorem 5.8, the price V_n of the Asian option at time n is some function v_n of S_n and Y_n, i.e.,

$$V_n = v_n(S_n, Y_n),\ \ n = 0, 1, \ldots, N.$$

Give a formula for $v_N(s, y)$, and provide an algorithm for computing $v_n(s, y)$ in terms of v_{n+1}.

Solution.

(i) We note that $\frac{S_{n+1}}{S_n}$ depends only on the $(n+1)$-st coin toss. Furthermore, $Y_{n+1} = Y_n + S_{n+1} = Y_n + S_n \cdot \frac{S_{n+1}}{S_n}$. The Independence Lemma implies that for any function h_{n+1}, we have

$$\begin{aligned} I\!E_n[h_{n+1}(S_{n+1}, Y_{n+1})] &= I\!E_n\left[h_{n+1}\left(S_n \cdot \frac{S_{n+1}}{S_n}, Y_n + S_n \cdot \frac{S_{n+1}}{S_n}\right)\right] \\ &= h_n(S_n, Y_n), \end{aligned}$$

where

$$\begin{aligned} h_n(s, y) &= I\!E h_{n+1}\left(s \cdot \frac{S_{n+1}}{S_n}, y + s \cdot \frac{S_{n+1}}{S_n}\right) \\ &= p h_{n+1}(us, y + us) + q h_{n+1}(ds, y + ds). \end{aligned}$$

Because $I\!E_n[h_{n+1}(S_{n+1}, Y_{n+1})]$ depends on the coin tosses only through S_n and Y_n, the two-dimensional process (S_n, Y_n), $n = 0, 1, \ldots, N$, is Markov.

(ii) We have $v_N(s,y) = f\left(\frac{y}{N+1}\right)$. To obtain the algorithm for v_n, we use the formula developed in (i) for h_n, putting in the discounting and using risk-neutral rather than actual probabilities. This results in

$$\begin{aligned} v_n(s,y) &= \frac{1}{1+r}\left[\tilde{p}v_{n+1}(us, y+us) + \tilde{q}v_{n+1}(ds, y+ds)\right], \\ & \qquad\qquad n = N-1, N-2, \dots, 0. \end{aligned}$$

Chapter 2, Exercise 13. In a binomial model, give an example of a stochastic process which is a martingale but is not Markov.

Solution. Let $M_0 = 0$ and

$$M_{n+1}(\omega_1 \dots \omega_n \omega_{n+1}) = \begin{cases} M_n(\omega \dots \omega_n) + 1 & \text{if } \omega_{n+1} = H, \\ M_n(\omega \dots \omega_n) - 1 & \text{if } \omega_{n+1} = T. \end{cases}$$

We assume that the probability for head on each toss is $p = \frac{1}{2}$, so the probability for tail is $q = \frac{1}{2}$. The process $M_0, M_1, \dots$ is called a *symmetric random walk*, and is easily seen to be a martingale.

Define the martingale transform $I_0, I_1, \dots$ by setting $I_0 = 0$ and

$$I_n = \sum_{j=0}^{n-1} \Delta_j(M_{j+1} - M_j), \ \ n = 1, 2, \dots.$$

Provided that each Δ_j depends only on the first j coin tosses, the martingale transform is a martingale (Exercise 6 of Chapter 2). Let us take $\Delta_0 = 0$, $\Delta_1(H) = 1$, $\Delta_1(T) = 0$. This builds a path dependence into the martingale transform, which destroys the Markov property.

To see this in detail, let us compute

$$\begin{aligned} &I_1(H) = \Delta_0(M_1(H) - M_0) = 0 \cdot 1 = 0, \\ &I_1(T) = \Delta_0(M_1(T) - M_0) = 0 \cdot (-1) = 0, \\ &I_2(HH) = I_1(H) + \Delta_1(H)(M_2(HH) - M_1(H)) = 0 + 1(2-1) = 1, \\ &I_2(HT) = I_1(H) + \Delta_1(H)(M_2(HT) - M_1(H)) = 0 + 1(0-1) = -1, \\ &I_2(TH) = I_1(T) + \Delta_1(T)(M_2(TH) - M_1(T)) = 0 + 0 \cdot (0-(-1)) = 0, \\ &I_2(TT) = I_1(T) + \Delta_1(T)(M_2(TT) - M_1(T)) = 0 + 0(-2-(-1)) = 0. \end{aligned}$$

The Markov property would require that for every function h_2, there is a function h_1 such that

$$\begin{aligned} \mathbb{E}[h_2(I_2)](H) &= h_1(I_1(H)) = h_1(0), \\ \mathbb{E}[h_2(I_2)](T) &= h_1(I_1(T)) = h_1(0). \end{aligned}$$

This would imply in turn that

$$\mathbb{E}[h_2(I_2)](H) = \mathbb{E}[h_2(I_2)](T).$$

But we can compute

$$\begin{aligned} \mathbb{E}[h_2(I_2)](H) &= \frac{1}{2}h_2(I_2(HH)) + \frac{1}{2}h_2(I_2(HT)) = \frac{1}{2}h_2(1) + \frac{1}{2}h_2(-1), \\ \mathbb{E}[h_2(I_2)](T) &= \frac{1}{2}h_2(I_2(TH)) + \frac{1}{2}h_2(I_2(TT)) = h_2(0). \end{aligned}$$

These two expressions are generally not the same. For example, if we take $h_2(x) = x^2$, then

$$\begin{aligned} \frac{1}{2}h_2(1) + \frac{1}{2}h_2(-1) &= 1, \\ h_2(0) &= 0. \end{aligned}$$

Therefore, this martingale transform is not Markov.

The point is that if at time one, you are told only the value of I_1, you have no information about the coin toss because $I_1(\omega_1) = 0$ regardless of the value of ω_1. However, the value of ω_1 is relevant to the computation $\mathbb{E}_2[h_2(I_2)](\omega_1)$.

Chapter 3, Exercise 2. Let $\mathbb{P}$ be a probability measure on a finite probability space Ω. Let Z be a random variable on Ω with the property that $\mathbb{P}(Z \geq 0) = 1$ and $\mathbb{E}Z = 1$. For $\omega \in \Omega$, define $\widetilde{\mathbb{P}}(\omega) = Z(\omega)\mathbb{P}(\omega)$, and for events $A \subset \Omega$, define $\widetilde{\mathbb{P}}(A) = \sum_{\omega \in A} \widetilde{\mathbb{P}}(\omega)$. Show the following.

(i) $\widetilde{\mathbb{P}}$ is a probability measure, i.e., $\widetilde{\mathbb{P}}(\Omega) = 1$.

(ii) If Y is a random variable, then $\widetilde{\mathbb{E}}Y = \mathbb{E}[ZY]$.

(iii) If A is an event with $\mathbb{P}(A) = 0$, then $\widetilde{\mathbb{P}}(A) = 0$.

(iv) Assume that $\mathbb{P}(Z > 0) = 1$. Show that if A is an event with $\widetilde{\mathbb{P}}(A) = 0$, then $\mathbb{P}(A) = 0$.

When two probability measures agree which events have probability zero, i.e., $\mathbb{P}(A) = 0$ if and only if $\widetilde{\mathbb{P}}(A) = 0$, the measures are said to be *equivalent*. From (iii) and (iv) above, we see that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent under the assumption that $\mathbb{P}(Z > 0) = 1$.

(v) Show that if $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, then they agree which events have probability one, i.e., $\mathbb{P}(A) = 1$ if and only if $\widetilde{\mathbb{P}}(A) = 1$.

(vi) Construct an example in which we have only $\mathbb{P}(Z \geq 0) = 1$ and $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are not equivalent.

In finance models, the risk-neutral probability measure and actual probability measure must always be equivalent. They agree about what is possible and what is impossible.

Solution.

(i) $\widetilde{\mathbb{P}}(\Omega) = \sum_{\omega \in \Omega} \widetilde{\mathbb{P}}(\omega) = \sum_{\omega \in \Omega} Z(\omega)\mathbb{P}(\omega) = \mathbb{E}Z = 1$.

(ii) $\widetilde{\mathbb{E}}Y = \sum_{\omega \in \Omega} Y(\omega)\widetilde{\mathbb{P}}(\omega) = \sum_{\omega \in \Omega} Y(\omega)Z(\omega)\mathbb{P}(\omega) = \mathbb{E}[YZ]$.

(iii) Assume $\mathbb{P}(A) = 0$. Because $\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega) = 0$, we have $\mathbb{P}(\omega) = 0$ for every $\omega \in A$. Therefore

$$\widetilde{\mathbb{P}}(A) = \sum_{\omega \in A} \widetilde{\mathbb{P}}(\omega) = \sum_{\omega \in A} Z(\omega)\mathbb{P}(\omega) = 0.$$

(iv) Let $B = \{\omega \in \Omega; Z(\omega) \leq 0\}$, so that $Z(\omega) > 0$ for every $\omega \in B^c$. By assumption, $\mathbb{P}(B) = 0$. Let A be an event with $\widetilde{\mathbb{P}}(A) = 0$. Then

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) = \mathbb{P}(A \cap B^c) = \sum_{\omega \in A \cap B^c} \mathbb{P}(\omega).$$

But for every $\omega \in A \cap B^c$, we have $Z(\omega)\mathbb{P}(\omega) = \widetilde{\mathbb{P}}(\omega) = 0$, and $Z(\omega) > 0$. Therefore, $\mathbb{P}(\omega) = 0$ for every $\omega \in A \cap B^c$. Thus $\mathbb{P}(A) = 0$.

(v) We assume that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent. Suppose $\mathbb{P}(A) = 1$. Then $\mathbb{P}(A^c) = 0$, so $\widetilde{\mathbb{P}}(A^c) = 0$, which implies $\widetilde{\mathbb{P}}(A) = 1$. Suppose $\widetilde{\mathbb{P}}(A) = 1$. Then $\widetilde{\mathbb{P}}(A^c) = 0$, so $\mathbb{P}(A^c) = 0$, which implies $\mathbb{P}(A) = 1$.

(vi) Let $\Omega = \{H, T\}$, $I\!P(H) = I\!P(T) = \frac{1}{2}$, $Z(H) = 2$, $Z(T) = 0$. Then $\widetilde{I\!P}(H) = 1$, $\widetilde{I\!P}(T) = 0$. The set $A = \{T\}$, has zero probability under $\widetilde{I\!P}$ but not under $I\!P$.

Chapter 3, Exercise 4. This problem refers to the model of Example 1.2, whose Radon-Nikodym process Z_n appears in Figure 2.1.

(i) Compute the state price densities

$$\begin{aligned}
&\zeta_3(HHH),\\
&\zeta_3(HHT) = \zeta_3(HTH) = \zeta_3(THH),\\
&\zeta_3(HTT) = \zeta_3(THT) = \zeta_3(TTH),\\
&\zeta_3(TTT)
\end{aligned}$$

explicitly.

(ii) Use the numbers computed in (i) in formula (1.9) to find the time-zero price of the Asian option of Exercise 7 of Chapter 1. You should get $v_0(4, 4)$ computed in part (ii) of that exercise.

(iii) Compute also the state price densities $\zeta_2(HT) = \zeta_2(TH)$.

(iv) Use the risk-neutral pricing formula (2.9) in the form

$$\begin{aligned}
V_2(HT) &= \frac{1}{\zeta_2(HT)} I\!E_2[\zeta_3 V_3](HT),\\
V_2(TH) &= \frac{1}{\zeta_2(TH)} I\!E_2[\zeta_3 V_3](TH)
\end{aligned}$$

to compute $V_2(HT)$ and $V_2(TH)$. You should get $V_2(HT) = v_2(4, 16)$ and $V_2(TH) = v_2(4, 10)$, where $v_2(s, y)$ was computed in part (ii) of Exercise 7 of Chapter 1. Note that $V_2(HT) \neq V_2(TH)$.

Solution.

(i) In general, $\zeta_3 = \left(\frac{4}{5}\right)^3 Z_3$. In particular,

$$\begin{aligned}
&\zeta_3(HHH) = 0.216,\\
&\zeta_3(HHT) = \zeta_3(HTH) = \zeta_3(THH) = 0.432,\\
&\zeta_3(HTT) = \zeta_3(THT) = \zeta_3(TTH) = 0.864,\\
&\zeta_3(TTT) = 1.728.
\end{aligned}$$

(ii) The Asian option payoff is

$$V_3(HHH) = 11,\ V_3(HHT) = 5,\ V_3(HTH) = 2,\ V_3(HTT) = 0.50,$$
$$V_3(THH) = 0.50,\ V_3(THT) = 0,\ V_3(TTH) = 0,\ V_3(TTT) = 0.$$

We compute

$$\begin{aligned}
V_0 &= \sum_{\omega\in\Omega} V_3(\omega)\zeta(\omega)I\!P(\omega)\\
&= 11\cdot 0.216\cdot\frac{8}{27} + 5\cdot 0.432\cdot\frac{4}{27} + 2\cdot 0.432\cdot\frac{4}{27}\\
&\qquad +0.50\cdot 0.864\cdot\frac{2}{27} + 0.50\cdot 0.432\cdot\frac{4}{27}\\
&= \frac{0.216}{27}(88+40+16+4+4)\\
&= 1.216.
\end{aligned}$$

(iii) In general, $\zeta_2 = \left(\frac{4}{5}\right)^2 Z_2$. In particular,

$$\zeta_2(HT) = \zeta_2(TH) = \frac{16}{25}\cdot\frac{9}{8} = 0.72.$$

(iv) We compute

$$\begin{aligned}
V_2(HT) &= \frac{1}{\zeta_2(HT)} I\!E_2[\zeta_3 V_3](HT)\\
&= \frac{1}{\zeta_2(HT)}[p\zeta_3(HTH)V_3(HTH)) + q\zeta_3(HTT)V_3(HTT)]\\
&= \frac{1}{0.72}\left[\frac{2}{3}\cdot 0.432\cdot 2 + \frac{1}{3}\cdot 0.864\cdot 0.50\right]\\
&= 1,\\
V_2(TH) &= \frac{1}{\zeta_2(TH)} I\!E_2[\zeta_3 V_3(TH)\\
&= \frac{1}{\zeta_2(TH)}[p\zeta_3(THH)V_3(THH) + q\zeta_3(THT)V_3(THT)]\\
&= \frac{1}{0.72}\left[\frac{2}{3}\cdot 0.432\cdot 0.50 + \frac{1}{3}\cdot 0.864\cdot 0\right]\\
&= 0.20.
\end{aligned}$$