46-679 Multi-Period Pricing Models
Homework Set 3
Due August 3, 1999
Solutions

**Exercise 5.** Toss a coin repeatedly. Assume the probability of head on each toss is $\frac{1}{2}$, as is the probability of tail. Let $X_j = 1$ if the $j$-th toss results in a head, and $X_j = -1$ if the $j$-th toss results in a tail. Consider the stochastic process $M_0, M_1, M_2, \ldots$ defined by $M_0 = 0$ and

$$M_n(\omega_1 \ldots \omega_n) = \sum_{j=1}^{n} X_j, \ n \geq 1.$$

This is called a *symmetric random walk*; with each head, it steps up one, and with each tail, it steps down one.

**(i)** Using the properties of Theorem 3.1, show that $M_0, M_1, M_2, \ldots$ is a martingale.

**(ii)** Let $\sigma$ be a positive constant, and for $n \geq 0$, define

$$S_n = e^{\sigma M_n} \left( \frac{2}{e^{\sigma} + e^{-\sigma}} \right)^n.$$

Show that $S_0, S_1, S_2, \ldots$ is a martingale.

**Solution.**

**(i)** Clearly , the process is adapted, so that $\mathbf{E}_n M_n = M_n$ for all n.

$$\begin{aligned} \mathbf{E}_n M_{n+1} &= \mathbf{E}_n(M_n + X_{n+1}) = \mathbf{E}_n M_n + \mathbf{E}_n X_{n+1} \\ &= M_n + \mathbf{E} X_{n+1} = M_n + 0 = M_n \end{aligned}$$

**(ii)** We have

$$\begin{aligned} \mathbf{E}_n S_{n+1} &= \mathbf{E}_n \left( e^{\sigma M_{n+1}} \left( \frac{2}{e^{\sigma} + e^{-\sigma}} \right)^{n+1} \right) \\ &= \left( \frac{2}{e^{\sigma} + e^{-\sigma}} \right)^{n+1} \mathbf{E}_n \left( e^{\sigma M_n} e^{\sigma X_{n+1}} \right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1} e^{\sigma M_n}\mathbf{E}_n\left(e^{\sigma X_{n+1}}\right)\\
&= \left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right) S_n\mathbf{E}\left(e^{\sigma X_{n+1}}\right)\\
&= \left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right) S_n\left(\frac{1}{2}e^{\sigma}+\frac{1}{2}e^{-\sigma}\right) = S_n
\end{aligned}
$$

**Exercise 6. (Discrete-time stochastic integral)** Suppose $M_0, M_1, \ldots,$ $M_N$ is a martingale, and let $\Delta_0, \Delta_1, \ldots, \Delta_{N-1}$ be an adapted process. Define the *discrete-time stochastic integral* (sometimes called a *martingale transform*) $I_0, I_1, \ldots, I_N$ by setting $I_0 = 0$ and

$$
I_n = \sum_{j=0}^{n-1} \Delta_j (M_{j+1} - M_j),\ n = 1, \ldots, N.
$$

Show that $I_0, I_1, \ldots, I_N$ is a martingale.
**Solution.** The process $I_n$ is adapted, as can be easily seen from the formula. In particular, we have $\mathbf{E}_n I_n = I_n$ for all n.

$$
\begin{aligned}
\mathbf{E}_n I_{n+1} &= \mathbf{E}_n\left[I_n + \Delta_n(M_{n+1} - M_n)\right]\\
&= \mathbf{E}_n I_n + \mathbf{E}_n\left[\Delta_n(M_{n+1} - M_n)\right]\\
&= I_n + \Delta_n \mathbf{E}_n(M_{n+1} - M_n)\\
&= I_n + \Delta_n(\mathbf{E}_n M_{n+1} - \mathbf{E}_n M_n)\\
&= I_n + \Delta_n(M_n - M_n) = I_n + 0 = I_n
\end{aligned}
$$

**Exercise 9. (Put-call parity)** In an $N$-period binomial model, consider a European call with payoff $C_N = (S_N - K)^+$ at time $N$. The price $C_n$ of this call at earlier times is given by the risk-neutral pricing formula (4.11):

$$
C_n = \widetilde{\mathbf{E}}_n\left[\frac{C_N}{(1+r)^{N-n}}\right],\ n = 0, 1, \ldots, N-1.
$$

Consider also a put with payoff $P_N = (K - S_N)^+$ at time $N$, whose price at earlier times is

$$
P_n = \widetilde{\mathbf{E}}_n\left[\frac{P_N}{(1+r)^{N-n}}\right],\ n = 0, 1, \ldots, N-1.
$$

Finally, consider a *forward contract* to buy one share of stock at time $N$ for $K$ dollars. The price of this contract at time $N$ is $F_N = S_N - K$, and its price at earlier times is

$$F_n = \widetilde{\mathbf{E}}_n \left[ \frac{F_N}{(1+r)^{N-n}} \right], \ n = 0, 1, \ldots, N-1.$$

(Note that unlike the call, the forward contract requires that the stock be purchased at time $N$ for $K$ dollars, and has a negative payoff if $S_N < K$.)

**(i)** If at time zero you buy a forward contract and a put, and hold them until expiration, explain why the payoff you receive is the same as the payoff of a call, i.e., explain why $C_N = F_N + P_N$.

**(ii)** Using the risk-neutral pricing formulas given above for $C_n$, $P_n$ and $F_n$ and the linearity of conditional expectations, show that $C_n = F_n + P_n$ for every $n$.

**(iii)** Using the fact that the discounted stock price is a martingale under the risk-neutral measure, show that $F_0 = S_0 - \frac{K}{(1+r)^N}$.

**(iv)** Suppose you begin at time zero with $F_0$, buy one share of stock, borrowing money as necessary to do that, and make no further trades. Show that at time $N$ you have a portfolio valued at $F_N$. (This is called a *static replication* of the forward contract. If you sell the forward contract for $F_0$ at time zero, you can use this static replication to hedge your short position in the forward contract.)

**(v)** The *forward price* of the stock at time zero is defined to be that value of $K$ which causes the forward contract to have price zero at time zero. The forward price in this model $(1+r)^N S_0$. Show that at time zero, the price of a call struck at the forward price is the same as the price of a put struck at the forward price. This fact is called *put-call parity*.

**(vi)** If we choose $K = (1+r)^N S_0$, we just saw in (v) that $C_0 = P_0$. Do we have $C_n = P_n$ for every $n$?

**Solution.**

**(i)** Consider three cases:

**case** $S_N = K$. Then $C_N = P_N = F_N = 0$;

**case** $S_N > K$. Then $P_N = 0$ and $C_N = S_N - K = F_N$;

**case** $S_N < K$. Then $C_N = 0$ and $P_N = K - S_N = -F_N$.

In all three cases, we see that $C_N = F_N + P_N$.

**(ii)**

$$\begin{aligned} C_n &= \widetilde{\mathbf{E}}_n\left[\frac{C_N}{(1+r)^{N-n}}\right] = \widetilde{\mathbf{E}}_n\left[\frac{F_N + P_N}{(1+r)^{N-n}}\right] \\ &= \widetilde{\mathbf{E}}_n\left[\frac{F_N}{(1+r)^{N-n}}\right] + \widetilde{\mathbf{E}}_n\left[\frac{P_N}{(1+r)^{N-n}}\right] = F_n + P_n \end{aligned}$$

**(iii)**

$$\begin{aligned} F_0 &= \widetilde{\mathbf{E}}\left[\frac{F_N}{(1+r)^N}\right] = \widetilde{\mathbf{E}}\left[\frac{S_N - K}{(1+r)^N}\right] \\ &= \widetilde{\mathbf{E}}\left[\frac{S_N}{(1+r)^N}\right] - \widetilde{\mathbf{E}}\left[\frac{K}{(1+r)^N}\right] = S_0 - \frac{K}{(1+r)^N} \end{aligned}$$

**(iv)** At time zero, our portfolio is

$$F_0 = S_0 + (F_0 - S_0).$$

At time $N$, the value of the portfolio is

$$\begin{aligned} & S_N + (1+r)^N(F_0 - S_0) \\ = \; & S_N + (1+r)^N\left(-\frac{K}{(1+r)^N}\right) \\ = \; & S_N - K = F_N. \end{aligned}$$

**(v)** First of all, if $K = (1+r)^N S_0$, then, by (iii), $F_0 = 0$. Further, if $F_0 = 0$, then, by (ii), $C_0 = F_0 + P_0 = P_0$.

**(vi)** No. This would mean, in particular, that $C_N = P_N$, and hence $(S_N - K)^+ = (K - S_N)^+$, which in turn would imply that $S_N(\omega) = K$ for all $\omega$, which most often is clearly not the case.

**Exercise 10. (Chooser option)** Let $1 \leq m \leq N-1$ and $K > 0$ be given. A *chooser option* is a contract sold at time zero which confers on its owner the right to receive either a call or a put at time $m$. The owner of the chooser may wait until time $m$ before choosing. The call or put chosen expires at time $N$ with strike price $K$. Show that the time-zero price of a chooser option is the sum of the time-zero price of a put, expiring at time $N$ and having strike price $K$, and a call, expiring at time $m$ and having strike price $\frac{K}{(1+r)^{N-m}}$.

**Solution.** According to the risk-neutral pricing formula (4.11), at time $m$, the price of the call is

$$\frac{1}{(1+r)^{N-m}}\widetilde{\mathbf{E}}_m(S_N - K)^+,$$

an the price of the put is

$$\frac{1}{(1+r)^{N-m}}\widetilde{\mathbf{E}}_m(K - S_N)^+.$$

The owner of the chooser will, of course, choose the "better deal", so the price of the chooser option at time $m$ is

$$\begin{aligned}
V_m &= \frac{1}{(1+r)^{N-m}} \max\left\{\widetilde{\mathbf{E}}_m(S_N - K)^+, \widetilde{\mathbf{E}}_m(K - S_N)^+\right\} \\
&= \frac{1}{(1+r)^{N-m}} \max\left\{\widetilde{\mathbf{E}}_m[(K - S_N)^+ + (S_N - K)], \widetilde{\mathbf{E}}_m(K - S_N)^+\right\} \\
&= \frac{1}{(1+r)^{N-m}}\widetilde{\mathbf{E}}_m(K - S_N)^+ + \frac{1}{(1+r)^{N-m}} \max\left\{\widetilde{\mathbf{E}}_m(S_N - K), 0\right\} \\
&= \frac{1}{(1+r)^{N-m}}\widetilde{\mathbf{E}}_m(K - S_N)^+ + \frac{1}{(1+r)^{N-m}} \left(\widetilde{\mathbf{E}}_m(S_N - K)\right)^+ \\
&= \frac{1}{(1+r)^{N-m}}\widetilde{\mathbf{E}}_m(K - S_N)^+ + \left(S_m - \frac{K}{(1+r)^{N-m}}\right)^+.
\end{aligned}$$

Consequently, the price of the chooser option at time zero is:

$$\begin{aligned}
V_0 &= \frac{1}{(1+r)^m}\widetilde{\mathbf{E}}V_m \\
&= \frac{1}{(1+r)^N}\widetilde{\mathbf{E}}(K - S_N)^+ + \frac{1}{(1+r)^m}\widetilde{\mathbf{E}}\left(S_m - \frac{K}{(1+r)^{N-m}}\right)^+.
\end{aligned}$$